

# Output feedback quantized observer-based synchronization of linear multi-agent systems over jointly connected topologies

Yang Meng<sup>1,2</sup>, Tao Li<sup>2,\*</sup>,† and Ji-Feng Zhang<sup>1</sup>

<sup>1</sup>Key Laboratory of Systems Control, Academy of Mathematics Systems Science, Chinese Academy of Sciences, Beijing 100190, China

<sup>2</sup>Shanghai Key Laboratory of Power Station Automation Technology, School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200072, China

## SUMMARY

The synchronization problem of linear over-actuated multi-agent systems with unmeasurable states is studied in this paper, under both limited communication data rate and switching topology flows. A class of adaptive quantized observer-based encoding–decoding schemes and a class of certainty equivalence principle-based control protocols are proposed. By developing the graph-based space decomposition technique and analyzing the closed-loop quantized dynamic equations, it is shown that if the network topology flow is jointly connected, the communication channels are periodic active, and the agent dynamics is observable, and with the orthogonal system matrix, the proposed communication and control protocols can ensure the closed-loop system to achieve synchronization exponentially fast with finite bits of information exchange per step. Copyright © 2015 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

The past few years have witnessed a rapid development of the coordination of multi-agent systems. The synchronization of agent networks, which means to drive all agents' states to the same, is a fundamental problem of distributed coordination and plays important roles in many areas, such as the cooperative control of unmanned air vehicles, the formation and flocking of robots, and so on. This problem has attracted lots of attention by the control community, and various synchronization protocols have been proposed (see [1–4] and the references therein). With digital communication channels, the information communication between adjacent agents is usually an integrated progress of quantization, encoding, transmitting, and decoding [5–7]. This communication mechanism requires only finite bits of information exchange per step and is convenient for implementation. So the study on quantized coordination with digital communication becomes a hot topic, which is of both theoretic and practical value.

There has been a number of researchers who devote themselves to the quantized synchronization of multi-agent systems. For single-integrator dynamics with fixed network topologies, the average-consensus and weighted average-consensus were considered in [5–8]. Frasca *et al.* [5] proposed a static quantized averaging algorithm for agents with real states and proved that all agents' states

\*Correspondence to: Tao Li, Shanghai Key Laboratory of Power Station Automation Technology, School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200072, China.

This work was accomplished when Yang Meng was a research assistant of School of Mechatronic Engineering and Automation, Shanghai University.

†E-mail: sixumuzi@shu.edu.cn

will enter a small neighborhood of the initial average in both the worst-case framework and the probabilistic framework. Dynamic quantization scheme was first studied in Carli *et al.* [6]. They designed an encoder and a set of decoders for each agent to send and receive information. By transforming the quantization errors of logarithm quantizers to multiplicative disturbances, they proved that the closed-loop system will achieve precise consensus. Li *et al.* [7] proposed a broadcasting-type encoding–decoding scheme with vanishing scaling functions and proved that the multi-agent system can achieve consensus exponentially with only one bit information exchange per step. This result highlights the constraint on data rate and is practical for communication channels with any given capacity. For the case with general directed communication topologies, Li *et al.* [8] proposed a dynamic uniform quantization-based protocol, which achieves weighted average consensus and showed that if each agent sends one bit information to each of its neighbors and another one bit information to itself, then the whole system will achieve consensus. The works [5–8] were further extended to the case with switching topologies in [9–11].

Because of some physical limitations, the agents may be modeled as systems with partially measurable, or even unmeasurable states. Ma and Zhang [12] and You and Xie [13] studied the synchronization of continuous-time and discrete-time linear systems, respectively. They present conditions on the network topology and agent dynamics to ensure the existence of admissible control protocols under precise communication. Hengster-Movric *et al.* studied synchronization of discrete-time linear systems over directed graphs in [14]. They proposed two methods that decouple the design of the synchronization gains from specific graph properties: one based on  $H_\infty$ -type and the other based on  $H_2$ -type Riccati design. Wen and Ugrinovskii [15] considered the distributed  $H_\infty$  leader-following tracking control for discrete-time systems with high-dimensional dynamic leaders. They proposed a feedback control law that uses a state estimator and gave sufficient conditions to enable all followers to track the leader while achieving the desired  $H_\infty$  tracking performance. Different from single-agent systems, the quantized synchronization of multi-agent systems with unmeasurable states requires each agent to observe not only its own but also its neighbors' states dynamically. This makes the state observation and coordination of multi-agent systems much more complicated. Besides the unmeasurable states and the limited channel capacity, another information constraint in multi-agent networks is the switching network topology due to link failures, packet dropouts, or high-level scheduling commands. For the case with switching topologies, the link failures of communication channels may lead to information mismatching between senders and receivers, so the broadcasting-type encoders and decoders proposed in [6] and [7] cannot be used. In a word, the problem on the coordination of multi-agent networks with an integrated information uncertainties due to unmeasurable states, switching topologies, and the data rate constraint is much more challenging.

In this paper, we consider the synchronization of linear over-actuated multi-agent systems with unmeasurable states, finite communication data rate, and switching topologies. We propose a class of quantized observer-based encoding–decoding communication schemes and a class of certainty equivalence principle-based control protocols using the inner states and outputs of encoders and decoders. To avoid the information inconsistency between the sender and the receiver, we use a channel activeness-based information updating rule to design the encoder and the decoder for each communication channel: the encoder and the decoder update their internal states simultaneously depending on whether the channel is active or not at the present time. We first consider the case with precise communication, full state feedback, and switching topologies. Motivated by [16], we develop the graph-based space decomposition technique and the Lyapunov method for discrete-time linear multi-agent systems. We establish the result that the joint connectivity of the network topology flow can ensure the exponential convergence of synchronization errors. Based on the aforementioned results, for the case with finite-level quantization, unmeasurable states, and switching topologies, we show that if the network topology flow is jointly connected, the communication channels are periodic active, and the agent dynamics is observable and with the orthogonal system matrix, then the proposed communication and control protocols will drive the multi-agent system to synchronization with finite bits of information exchange per step.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and formulate the problem to be studied. In Section 3, we first consider the case of precise communication, and

then the finite-level quantized communication and give the main results of this paper. In Section 4, we give a numerical example to verify the effectiveness of our results. The concluding remarks and some future topics are given in Section 5.

The following notations will be used in this paper. Denote the column vectors or matrices with all elements being 1 by  $\mathbf{1}$ . Denote the identity matrix with dimension  $n$  by  $I_n$ . Denote the sets of real numbers, conjugate numbers, and nonnegative integers by  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{N}$ , respectively, and  $\mathbb{R}^n$  denotes the  $n$ -dimensional real space. For a vector  $X \in \mathbb{R}^n$  or a matrix  $X = [x_{ij}] \in \mathbb{R}^{n \times m}$ , its transpose is denoted by  $X^H$ , and its Euclidean norm and the infinite norm are denoted by  $\|X\|$  and  $\|X\|_\infty$ , respectively. The spectral radius of a matrix  $X$  is denoted by  $\rho(X)$ . The Kronecker product, denoted by  $\otimes$ , facilitates the manipulation of matrices by the following properties: (i)  $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$  and (ii)  $(A \otimes B)^H = A^H \otimes B^H$ . Denote the block diagonal matrix with its diagonal matrices  $A_1, \dots, A_k$  by  $\text{diag}(A_1, \dots, A_k)$ .  $\mathcal{B}_r = \{X \in \mathbb{R}^n \mid \|X\|_\infty < r\}$  represents an open ball with radius  $r$  in  $\mathbb{R}^n$ . The angle between two vectors  $y \in \mathbb{R}^n$  and  $z \in \mathbb{R}^n$  is defined as  $\theta_{y,z} = \arccos(y^H z / \|y\| \|z\|)$  where  $\|z\| = z^H z$ . If  $y^H z = 0$ , then  $y$  and  $z$  are called orthogonal, denoted by  $y \perp z$ .  $S \subseteq \mathbb{R}^n$ , and  $S' \subseteq \mathbb{R}^n$  are two subspaces of  $\mathbb{R}^n$ . If  $s^H s' = 0$  for any  $s \in S$  and  $s' \in S'$ , then the two subspaces are orthogonal, denoted by  $S \perp S'$ . The space  $S^* = S + S'$  with  $S \perp S'$  is denoted by  $S^* = S \oplus S'$ . If  $\mathbb{R}^n = S \oplus S'$ , then  $S'$  is called the orthogonal complement of  $S$ . The angle between two subspaces  $S$  and  $S'$  is defined as  $\theta_{S,S'} = \min_{s \in S, s' \in S'} \theta_{s,s'}$ . It can be seen that  $0 \leq \theta_{S,S'} \leq \pi/2$ . Consider two positive semi-definite matrices  $G_1$  and  $G_2$ . Denote  $\text{rank}(G_1) = r_1$ ,  $\text{rank}(G_1 + G_2) = r_{1,2}$ .

## 2. PRELIMINARIES

For a multi-agent system consisting of  $N$  agents, the communication structures among different agents are represented by a sequence of undirected graphs  $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$ ,  $t \in \mathbb{N}$ , where  $\mathcal{V} = \{1, \dots, N\}$  is the node set with each node representing an agent, and  $\mathcal{E}(t)$  is the edge set. An edge  $(j, i) \in \mathcal{E}(t)$  if and only if there is a communication channel from  $j$  to  $i$  at time  $t$ . Then, agent  $i$  is called the receiver, and agent  $j$  is called the sender, or  $i$ 's neighbor. The set of agent  $i$ 's neighbors is denoted by  $N_i(t) = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}(t)\}$ ,  $t \in \mathbb{N}$ . Denote  $\mathcal{N}_i = \bigcap_{k=1}^{\infty} \bigcup_{t=k}^{\infty} N_i(t)$ . By its definition, each element of  $\mathcal{N}_i$  is agent  $i$ 's neighbor for infinite times. In this paper, we assume that if an edge is active at some time, then it will be active for infinite times, that is,

$$(A1) \quad \mathcal{N}_i = \bigcup_{t=1}^{\infty} N_i(t).$$

We denote  $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N \times N}$  as the adjacent matrix of  $\mathcal{G}(t)$ , where  $a_{ij}(t) = 1$  if  $j \in N_i(t)$  and  $a_{ij}(t) = 0$  otherwise. Here, we assume that  $a_{ii}(t) = 0$ ,  $i \in \mathcal{V}$ ,  $t \in \mathbb{N}$ . For any  $t \in \mathbb{N}$ , since  $\mathcal{G}(t)$  is undirected,  $\mathcal{A}^H(t) = \mathcal{A}(t)$ . Denote  $\text{deg}_i^{\text{in}}(t) = \sum_{j=1}^N a_{ij}(t)$ ,  $\text{deg}_i^{\text{out}}(t) = \sum_{j=1}^N a_{ji}(t)$  as the in-degree and out-degree of node  $i$ , and  $\mathcal{D}(t) = \text{diag}(\text{deg}_1^{\text{in}}(t), \dots, \text{deg}_N^{\text{in}}(t))$  as the degree matrix of  $\mathcal{G}(t)$ . The Laplacian matrix  $\mathcal{L}(t)$  of  $\mathcal{G}(t)$  is defined as  $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$ , and its eigenvalues in an ascending order are denoted by  $\lambda_1(\mathcal{L}(t)) = 0$ ,  $\lambda_i(\mathcal{L}(t))$ ,  $i = 2, \dots, N$ . A sequence of edges  $(i_1, i_2) \in \mathcal{G}(t)$ ,  $(i_2, i_3) \in \mathcal{G}(t)$ ,  $\dots$ ,  $(i_{k-1}, i_k) \in \mathcal{G}(t)$  is called a path from  $i_1$  to  $i_k$  at  $t$ . The union graph for a sequence of graphs  $\mathcal{G}_1, \dots, \mathcal{G}_l$ , each of which has the same vertex set  $\mathcal{V}$ , is denoted by  $\mathcal{G}_{1,\dots,l}$  with vertex set  $\mathcal{V}$  and edge set being the union of the edge sets of all graphs in the series. The Laplacian matrix of the union graph  $\mathcal{G}_{1,\dots,l}$  is defined as  $\sum_{i=1}^l \mathcal{L}_i$ . The sequence  $\mathcal{G}_1, \dots, \mathcal{G}_l$  is called jointly connected if the union graph  $\mathcal{G}_{1,\dots,l}$  is connected.

## 3. MAIN RESULTS

### 3.1. Case with precise communication and full state feedback

In this section, we assume that the inter-agent communication is precise, and the states of each agent are fully measurable. We will show that the systems can achieve synchronization exponentially fast under jointly connected topologies. It is shown that the period of the joint connectivity plays an important role in obtaining the exponential convergence speed.

Consider the following multi-agent system with  $N$  agents, each of which following

$$x_i(t + 1) = Ax_i(t) + Bu_i(t), \quad i = 1, \dots, N, \tag{1}$$

where  $x_i(t) \in \mathbb{R}^n$  and  $u_i(t) \in \mathbb{R}^m$  are the state and control input of agent  $i$ , respectively. For networked multi-agent systems, a control protocol is called distributed or neighbor-based, if the control input of each agent depends only on the information of its own and its neighbors. Denote  $X(t) = (x_1^H(t), \dots, x_N^H(t))^H$ ,  $\bar{X}(t) = \frac{1}{N} (\mathbf{1}_N^H \otimes I_n) X(t)$ , and  $\delta(t) = X(t) - (\frac{1}{N} \mathbf{1}_N \mathbf{1}_N^H \otimes I_n) X(t)$ . We say that the multi-agent system achieves synchronization if there is a distributed control protocol such that  $\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0, i, j = 1, \dots, N$ , or equivalently,  $\lim_{t \rightarrow \infty} \|\delta(t)\| = 0$ .

The communication topology flow is a sequence of undirected graphs  $\mathcal{G}(t), t \in \mathbb{N}$ . Because the number of nodes in each  $\mathcal{G}(t)$  is finite, the communication topologies always switching in a finite set  $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_M\}$ , which is denoted by  $\mathcal{G} = \{\mathcal{G}_i | i \in \mathcal{P}\}$ , where  $\mathcal{P} = \{1, \dots, M\}$ .

For any  $\mathcal{G}_i \in \mathcal{G}, i \in \mathcal{P}$ , denote by  $\mathcal{L}_i$ , the Laplacian matrix of  $\mathcal{G}_i$ . Here, we take the following control protocol:

$$u_i(t) = K \sum_{j=1}^N a_{ij}(t)(x_j(t) - x_i(t)), \quad i = 1, \dots, N. \tag{2}$$

We make the following assumptions:

- (A2) The system matrix  $A$  is orthogonal and  $B$  is full of row rank.
- (A3) There exists a constant  $T \geq 1$  and a sequence of intervals  $[t_k, t_{k+1})$ , where  $t_0 = 0$  and  $t_{k+1} - t_k \leq T, k \in \mathbb{N}$  such that  $\mathcal{G}(t), t \in [t_k, t_{k+1})$  are jointly connected,  $k \in \mathbb{N}$ .

*Remark 1*

For neutrally stable agents, there is a non-singular matrix  $O$  such that  $\tilde{A} = O^{-1}AO$  is a orthogonal matrix [17]. Let  $\tilde{x}_i(t) = O^{-1}x_i(t)$ , then by (1), we have  $\tilde{x}_i(t + 1) = \tilde{A}\tilde{x}_i(t) + \tilde{B}u_i(t)$  where  $\tilde{A}$  is orthogonal and  $\tilde{B} = O^{-1}B$ .

For each interval  $[t_k, t_{k+1})$ , there is a sequence of subintervals  $[t_{k_0}, t_{k_1}), \dots, [t_{k_{p-1}}, t_{k_p}), \dots, [t_{k_{m_k-1}}, t_{k_{m_k}})$ , with  $t_k = t_{k_0}$  and  $t_{k+1} = t_{k_{m_k}}$ , where  $t_{k_p} - t_{k_{p-1}} \geq 1, 1 \leq p \leq m_k$ , and during each subinterval, the communication topology does not change. It is obvious that  $m_k \leq T$ , so there is at most  $T$  subintervals in each interval. In this section, we take the control gain  $K = \mu B^H A$ , where  $\mu > 0$  is constant to be designed. By (1) and (2), we have

$$\delta(t + 1) = (I_N \otimes A - \mu \mathcal{L}(t) \otimes BB^H A) \delta(t). \tag{3}$$

Let  $V(\delta(t)) = \delta^H(t)\delta(t) = \sum_{i=1}^N \delta_i^H(t)\delta_i(t)$  be the Lyapunov function, and denoted by  $V(\delta)$  when not emphasizing on time. It can be seen that for any orthogonal matrix  $U \in \mathbb{R}^{N \times N}$  and  $\tilde{\delta} = (U \otimes I_n)\delta$ , we have  $V(\tilde{\delta}) = V(\delta)$ . For any given Laplacian matrix  $\mathcal{L}_p, p \in \mathcal{P}$  of rank  $r_p$ , there is an orthogonal matrix  $U_p = (e_1^p, \dots, e_{r_p}^p)$  such that

$$U_p^H \mathcal{L}_p U_p = \begin{pmatrix} \Lambda_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where  $\Lambda_p = \text{diag}(\lambda_1(\mathcal{L}_p), \dots, \lambda_{r_p}(\mathcal{L}_p))$  with  $\lambda_i(\mathcal{L}_p)$  denoting the nonzero eigenvalue of  $\mathcal{L}_p, i = 1, \dots, r_p$ . Denote  $\lambda_{min}$  as the smallest eigenvalue of  $A^H BB^H A$ . By Assumption A2, we can see that  $\lambda_{min} > 0$ . Denote  $\underline{\lambda} = \min_{i \in \{1, \dots, r_p\}, p \in \mathcal{P}} \lambda_i(\mathcal{L}_p)$ . Let  $\delta^p(t) = (U_p^H \otimes I_n)\delta(t)$ , so  $\delta_i^p(t) = ((e_i^p)^H \otimes I_n)\delta(t), i = 1, \dots, N$ , and  $V(\delta(t)) = V(\delta^p(t)) = \sum_{i=1}^N (\delta_i^p(t))^H \delta_i^p(t)$ . We have the following lemma.

*Lemma 3.1*

For the class of switching systems (3) and any interval  $[t_k, t_{k+1})$ , let  $\mathcal{L}_p$ , with a slight abuse of notations, be the Laplacian matrix associated with the subinterval  $[t_{k_{p-1}}, t_{k_p}), p = 1, \dots, m_k$ . If A2 holds and  $0 < \mu < \min_{p \in \mathcal{P}} \frac{1}{\|\mathcal{L}_p \otimes (A^H BB^H A)\|}$ , then we have

$$V(\delta(t_{k_p})) \leq V(\delta(t_{k_{p-1}})). \tag{4}$$

Furthermore,

$$\begin{aligned} (\delta_i^p(t_{k_p}))^H \delta_i^p(t_{k_p}) &\leq (\delta_i^p(t_{k_{p-1}}))^H \delta_i^p(t_{k_{p-1}}), \quad i = 1, \dots, N, \\ (\delta_i^p(t_{k_p}))^H \delta_i^p(t_{k_p}) &\leq \gamma_0 (\delta_i^p(t_{k_{p-1}}))^H \delta_i^p(t_{k_{p-1}}), \quad i = 1, \dots, r_p, \end{aligned}$$

where  $0 < \gamma_0 < 1$  is a constant independent of  $k$  and  $k_p$ .

*Proof*

Consider any  $t \in [t_{k_{p-1}}, t_{k_p})$ . By the definition of  $\delta^p(t)$ , we have  $\delta_i^p(t) = ((e_i^p)^H \otimes I_n) \delta(t)$ . According to (3), we know that

$$\delta_i^p(t + 1) = (A - \mu \lambda_i(\mathcal{L}_p) BB^H A) \delta_i^p(t),$$

which gives

$$(\delta_i^p(t + 1))^H \delta_i^p(t + 1) = (\delta_i^p(t))^H (I_n - 2\mu \lambda_i(\mathcal{L}_p) A^H BB^H A + \mu^2 \lambda_i^2(\mathcal{L}_p) (A^H BB^H A)^2) \delta_i^p(t). \tag{5}$$

Since  $0 < \mu < \min_{p \in \mathcal{P}} \frac{1}{\|\mathcal{L}_p \otimes (A^H BB^H A)\|}$ , we have  $\mu \lambda_i(\mathcal{L}_p) A^H BB^H A < I_n$  for any  $p \in \mathcal{P}, i = 1, \dots, r_p$ . Then, by (5), we have

$$\begin{aligned} (\delta_i^p(t + 1))^H \delta_i^p(t + 1) &\leq (\delta_i^p(t))^H (I_n - \mu \lambda_i(\mathcal{L}_p) A^H BB^H A) \delta_i^p(t) \\ &\leq (\delta_i^p(t))^H (I_n - \mu \lambda_i(\mathcal{L}_p) \lambda_{\min} I_n) \delta_i^p(t) \\ &= (1 - \mu \lambda_i(\mathcal{L}_p) \lambda_{\min}) (\delta_i^p(t))^H \delta_i^p(t). \end{aligned} \tag{6}$$

Let  $\gamma_0 = 1 - \mu \lambda_{\min}$ . By the definition of  $\mu$ , we can see that  $0 < \gamma_0 < 1$ . Noting that  $\lambda_i(\mathcal{L}_p) = 0, i = r_p + 1, \dots, N$ , then from (5) and (6), we have

$$\begin{aligned} (\delta_i^p(t + 1))^H \delta_i^p(t + 1) &= (\delta_i^p(t))^H \delta_i^p(t), \quad i = r_p + 1, \dots, N, \\ (\delta_i^p(t + 1))^H \delta_i^p(t + 1) &\leq \gamma_0 (\delta_i^p(t))^H \delta_i^p(t), \quad i = 1, \dots, r_p, \end{aligned}$$

which imply the conclusion of the lemma. □

*Remark 2*

Let  $V_{p,0}(\delta^p) = \sum_{i=r_p+1}^N (\delta_i^p)^H \delta_i^p$  and  $V_{p,\perp}(\delta^p) = \sum_{i=1}^{r_p} (\delta_i^p)^H \delta_i^p$ . According to Lemma 3.1, we have

$$\begin{aligned} V_{p,0}(\delta^p(t_{k_p})) &\leq V_{p,0}(\delta^p(t_{k_{p-1}})), \\ V_{p,\perp}(\delta^p(t_{k_p})) &\leq \gamma_0 V_{p,\perp}(\delta^p(t_{k_{p-1}})). \end{aligned} \tag{7}$$

Now, we have the main result of this section.

*Lemma 3.2*

For the linear multi-agent system (1), if **A2** and **A3** hold, then it can achieve synchronization exponentially fast under the control protocol (2), where  $K = \mu B^H A$  and  $0 < \mu < \min_{i \in \mathcal{P}} \frac{1}{\|\mathcal{L}_i \otimes (A^H BB^H A)\|}$ , that is, there exist  $0 < \beta < 1$  and  $C_0 > 0$ , which can be given explicitly according to the agent dynamics and the network topology flow, such that  $\|\delta(t)\| \leq C_0 \|\delta(0)\| \beta^t$ .

The proof of Lemma 3.2 is put in the Appendix.

*Remark 3*

Assumption A2 assumes that the input matrix  $B$  is full of row rank, which means that the agent dynamics is over-actuated. By this property, the strict decreasing of the Laypunov function can be guaranteed in Lemma 3.2. It is an interesting topic for future investigation that whether our results can be extended for underactuated multi-agent systems. Su and Huang [17] gave a useful clue for this direction, where they used an asymptotic analysis method to show the convergence of

the synchronization error without requiring that the input matrix is full of row rank. However, the method in [17] cannot give the convergence speed. Different from [17], here, the convergence speed of the synchronization error equation plays a key role in analyzing the overall closed-loop system due to the coupling between the synchronization error and state estimation error. Thus, the method in [17] is not available in this paper.

3.2. Case with finite-level quantized communication and output feedback

In what follows, we consider the case with finite-level quantized communication and unmeasurable agent states. We propose a class of adaptive quantized observer-based encoding–decoding schemes and a class of certainty equivalence principle-based control protocols. Here, the quantized observer based encoding–decoding schemes proposed by [9] and [18] are combined and extended for linear multi-agent systems with unmeasurable states and switching topologies. We also give the closed-loop analysis and show the exponential convergence speed.

3.2.1. Protocol design. Consider a multi-agent system consisting of  $N$  agents, each of which satisfies

$$\begin{cases} x_i(t + 1) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \tag{8}$$

where  $x_i(t) \in R^n, u_i(t) \in R^m, y_i(t) \in R^p$  are the state, the input, and the output of agent  $i$ , respectively. The communication topology flow is still represented by  $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}, t \in \mathbb{N}$ . The agent dynamics (8) together with the communication topology flow  $\{\mathcal{G}(t), t \in \mathbb{N}\}$  is called a dynamic network [3], which is denoted by  $(A, B, C, \mathcal{G}(t))$ .

Here, the communication channel between each pair of adjacent agents is digital and has finite bandwidth. So real-valued data should be quantized into finite symbols before transmitting. Under the switching topology flow, the broadcasting-type communication scheme may fail due to information mismatch between the sender and the receiver. Here, we use the channel activeness-based method [9] to realize the information communication: for each channel  $(j, i), i \in \mathcal{V}, j \in \mathcal{N}_i$ , if it is active, the sender  $j$  quantizes and encodes its output information through an encoder and sends it to the receiver  $i$ . After receiving the information,  $i$  estimates  $j$ 's state by a decoder. The encoder  $\Theta_{ji}$  maintained by agent  $j$  is given by

$$\Theta_{ji}(t) = \begin{cases} \xi_{ji}(0) = 0, \hat{u}_{ji}(0) = 0, \\ \hat{u}_{ji}(t - 1) = \hat{u}_{ji}(t - 2) + a_{ij}(t)g_u(t - 1)s_{u,ji}(t - 1), \\ \xi_{ji}(t) = A\xi_{ji}(t - 1) + a_{ij}(t)g(t - 1)Gs_{ji}(t - 1) + B\hat{u}_{ji}(t - 1), \\ s_{ji}(t - 1) = Q_{t-1}^{ji} \left( \frac{y_j(t - 1) - C\xi_{ji}(t - 1)}{g(t - 1)} \right), \\ s_{u,ji}(t - 1) = Q_{t-1}^{u,ji} \left( \frac{u_j(t - 1) - \hat{u}_{ji}(t - 2)}{g_u(t - 1)} \right). \end{cases} \tag{9}$$

Here,  $G$  is a parameter matrix to be designed.  $(\xi_{ji}^H(t), \hat{u}_{ji}^H(t - 1))^H$  is the inner state of  $\Theta_{ji}(t)$ , and  $(s_{ji}^H(t - 1), s_{u,ji}^H(t - 1))^H$  is the output of it.  $g(t)$  and  $g_u(t)$  are scaling functions. Here, we take  $g(t) = g_0\gamma^t$  and  $g_u(t) = l_0\gamma^t$ , where  $g_0, l_0$  and  $0 < \gamma < 1$  are parameters to be designed.

The decoder  $\Psi_{ji}$  maintained by agent  $i$  is given by

$$\Psi_{ji}(t) = \begin{cases} \hat{x}_{ji}(0) = 0, \tilde{u}_{ji}(0) = 0, \\ \tilde{u}_{ji}(t - 1) = \tilde{u}_{ji}(t - 2) + a_{ij}(t)g_u(t - 1)s_{u,ji}(t - 1), \\ \hat{x}_{ji}(t) = A\hat{x}_{ji}(t - 1) + a_{ij}(t)g(t - 1)Gs_{ji}(t - 1) + B\tilde{u}_{ji}(t - 1), \end{cases} \tag{10}$$

where  $\hat{x}_{ji}(t)$  is the output of  $\Psi_{ji}$ , which represents the estimation of  $j$ 's state. From (9) and (10), it can be seen that  $\hat{x}_{ji}(t) = \xi_{ji}(t)$  and  $\tilde{u}_{ji}(t - 1) = \hat{u}_{ji}(t - 1)$ . Denote  $E_{ji}(t) = \xi_{ji}(t) - x_j(t)$  and  $H_{ji}(t) = u_j(t) - \hat{u}_{ji}(t)$  as the estimation errors.

Here,  $Q_t^{ji}(\cdot)$  and  $Q_t^{u,ji}(\cdot)$  are time-varying uniform quantizers:

$$Q_t^{ji}(y) = \begin{cases} 0, & -\frac{1}{2} \leq y < \frac{1}{2}, \\ l, & l - \frac{1}{2} \leq y < l + \frac{1}{2}, \quad l = 1, \dots, L_{ji}(t) - 1, \\ L_{ji}(t), & y \geq L_{ji}(t) - \frac{1}{2}, \\ -Q_t^{ji}(-y), & y < -\frac{1}{2}, \end{cases}$$

$$Q_t^{u,ji}(y) = \begin{cases} 0, & -\frac{1}{2} \leq y < \frac{1}{2}, \\ l, & l - \frac{1}{2} \leq y < l + \frac{1}{2}, \quad l = 1, \dots, L_{u,ji}(t) - 1, \\ L_{u,ji}(t), & y \geq L_{u,ji}(t) - \frac{1}{2}, \\ -Q_t^{u,ji}(-y), & y < -\frac{1}{2}, \end{cases}$$

where  $2L_{ji}(t) + 1$  and  $2L_{u,ji}(t) + 1$  are quantization levels. If  $y \in R^n$ , then let  $Q_t^{ji}(y) = (Q_t^{ji}(y_1), \dots, Q_t^{ji}(y_n))^H$  and  $Q_t^{u,ji}(y) = (Q_t^{u,ji}(y_1), \dots, Q_t^{u,ji}(y_n))^H$ . Denote  $\Delta_{ji}(t) = \frac{y_j(t) - C\xi_{ji}(t)}{g(t)} - s_{ji}(t)$ ,  $W_{ji}(t) = u_j(t) - \hat{u}_{ji}(t - 1)$ ,  $\Delta_{u,ji}(t) = s_{u,ji}(t) - \frac{W_{ji}(t)}{g_u(t)}$ . Thus,  $\Delta_{ji}(t)$  and  $\Delta_{u,ji}(t)$  are quantization errors of  $Q_t^{ji}(\cdot)$  and  $Q_t^{u,ji}(\cdot)$ .

Based on the aforementioned communication protocol, we propose the following certainty equivalence principle-based admissible control protocol set:

$$\mathcal{U} = \left\{ u_i(t), i \in \mathcal{V}, t \in \mathbb{N} \mid u_i(t) = K \sum_{j \in \mathcal{N}_i} a_{ij}(t)(\hat{x}_{ji}(t) - \xi_{ij}(t)) \right\}, \tag{11}$$

where  $K$  is the control gain to be designed.

*Remark 4*

Here, we only design encoders and decoders for each  $(j, i)$ ,  $j \in \mathcal{N}_i$ ,  $i \in \mathcal{V}$ . According to **A1**, we have  $u_i(t) = K \sum_{j \in \mathcal{N}_i} a_{ij}(t)(\hat{x}_{ji}(t) - \xi_{ij}(t)) = K \sum_{j=1}^N a_{ij}(t)(\hat{x}_{ji}(t) - \xi_{ij}(t))$ . The idea is similar to the certainty equivalence principle. We use the estimates of the agents' states instead of the real states to construct the control protocol in (11). The effectiveness of the protocol will be shown in the succeeding text.

**3.2.2. Convergence analysis.** Next, we give the main result of this paper. For each channel  $(j, i)$ ,  $j \in \mathcal{N}_i$ ,  $i \in \mathcal{V}$ , denote  $0 = t_0^{ji}, \dots, t_l^{ji}, \dots$ , as the switching times of  $(j, i)$  such that it switches between inactivity and activity, at each  $t_l^{ji}$ ,  $l \in \mathbb{N}$ . Thus, on each interval  $[t_l^{ji}, t_{l+1}^{ji})$ ,  $a_{ij}(t) \equiv 1$  or  $0$ ,  $l \in \mathbb{N}$ . We need the following assumptions:

- (A4)**  $(A, C)$  is observable.
- (A5)** For any channel  $(j, i)$ ,  $j \in \mathcal{N}_i$ ,  $i \in \mathcal{V}$ , there exist two known constants  $\tau_1^{ji}$  and  $T_1^{ji}$  such that for any interval  $[t_l^{ji}, t_{l+1}^{ji})$ ,  $l \in \mathbb{N}$ , if  $(j, i)$  is active on it, then  $t_{l+1}^{ji} - t_l^{ji} = \tau_1^{ji}$ . If  $(j, i)$  is inactive on it, then  $t_{l+1}^{ji} - t_l^{ji} = T_1^{ji}$ .
- (A6)** There exist two known constants  $C_x$  and  $C_\delta$  such that

$$\max_{i \in \mathcal{V}} \|x_i(0)\|_\infty \leq C_x, \max_{i \in \mathcal{V}} \|\delta_i(0)\|_\infty \leq C_\delta.$$

*Remark 5*

Because the communication topologies are undirected graphs, it is easy to see that  $\tau_1^{ij} = \tau_1^{ji}$  and  $T_1^{ij} = T_1^{ji}$ . Denote  $d^* = \sup_{t \in \mathbb{N}} \max_{i \in \mathcal{V}} d_i(t)$ . It is obvious that  $d^* \leq N - 1$ .

*Lemma 3.3*

Let  $\beta(x) = (1 - \sigma^{2T} + \sigma^{2T}(1 - x\underline{\lambda}\lambda_{min}))^{1/2T}$  and  $C_0(x) = (1 - \sigma^{2T} + \sigma^{2T}(1 - x\underline{\lambda}\lambda_{min}))^{-1/2}$ , where  $x \in \mathbb{R}$ ,  $0 < \sigma < 1$  is a constant. For a class of linear switching systems,

$$x(t + 1) = (I_N \otimes A - \mathcal{L}(t) \otimes BK)x(t), \tag{12}$$

where  $x(t) \in \mathbb{R}^{nN}$  and  $(\mathbf{1}_N^H \otimes I_n)x(0) = 0$ ,  $K \in R^{m \times n}$  is the gain matrix to be designed.  $\mathcal{L}(t)$ ,  $t \in \mathbb{N}$  are Laplacian matrices of a sequence of undirected graphs with  $N$  nodes. Assume that **A2** and **A3** hold. Take  $K = \mu B^H A$ , where  $0 < \mu < \min_{p \in \mathcal{P}} \frac{1}{\|\mathcal{L}_p \otimes (A^H B B^H A)\|}$ , then the system (12) is exponential stable and  $\|x(t)\| \leq \beta^t(\mu)C_0(\mu)\|x(0)\|$ .

The proof of Lemma 3.3 is straightforward by using Lemma 3.2 and is omitted here.

Denote  $\tau_1 = \min_{i,j} \tau_1^{ji}$ ,  $\tau_2 = \max_{i,j} \tau_1^{ji}$ , and  $T_1 = \max_{i,j} T_1^{ji}$ . Denote  $\eta_0 = \max\{\rho(A) + \varepsilon_0\|A\|, 1\}$ , where  $\varepsilon_0 > 0$  is an arbitrary constant. Denote  $M_0 = \sqrt{n}(1 + 2/\varepsilon_0)^{n-1}$ . Denote  $\eta_1(X) = \rho(A - XC) + \varepsilon_1(X)\|A - XC\|$ , where  $X \in \mathbb{R}^{n \times p}$  and  $\varepsilon_1(X) = \frac{\rho(A - XC)}{\|A - XC\|}$ . Denote  $M_1(X) = \sqrt{n}(1 + 2/\varepsilon_1(X))^{n-1}$ ,  $R_0 = M_0\eta_0^{T_1}$ , and  $\rho_2(X) = M_1(X)\eta_1^{T_1}(X)R_0$ . For any  $x, y \in \mathbb{R}$ , denote

$$H(x, y) = 4\sqrt{nN}d^*x^2\|B^H A\|_\infty\|BB^H A\|C_0(x) + 2d^*x\|B^H A\|_\infty(y - \beta(x))C_0(x). \tag{13}$$

Now, we give the main theorem.

*Theorem 3.1*

For the dynamic network  $(A, B, C, \mathcal{G}(t))$ , assume that **A1–A6** hold and  $\tau_1 > n - 1$ . Select the parameters of the communication and control protocols as in the succeeding text.

- (i) Take  $G \in \{X \in \mathbb{R}^{n \times p} | \eta_1(X) < \min(1, (1/M_1(X)R_0)^{1/\tau_1})\}$ .
- (ii) Take  $(\mu, \gamma) \in \{(x, y) \in \mathbb{R}^2 | x \in (0, \min_{p \in \mathcal{P}} 1/\|\mathcal{L}_p \otimes (A^H B B^H A)\|), y \in (\max\{1/2, \eta_1, \beta(x), \rho_2^{1/(T_1+\tau_2)}\}, 1), a(x, y) < 1, b(x, y) < 1\}$ , where

$$\begin{aligned} a(x, y) &= \frac{(y + 1)\tilde{H}H(x, y)}{l_0y^{T_1+1}y^{T_1+\tau_2}(\eta_0 - y)(y^{T_1+\tau_2} - \rho_2)(y - \beta(x))}, \\ b(x, y) &= \frac{(y + 1)\tilde{H}H(x, y)}{(2l_0y - l_0)y^{T_1}y^{T_1+\tau_2}(y^{T_1+\tau_2} - \rho_2)(\eta_0 - 1)(y - \beta(x))}, \end{aligned} \tag{14}$$

$\tilde{H} = (M_1R_02^{T_1}l_0\sqrt{m}(\rho_2 + 1)\|B\|)$ ,  $\varepsilon_1, \eta_1, \rho_2$  and  $M_1$  are abbreviations of  $\varepsilon_1(G), \eta_1(G), \rho_2(G)$  and  $M_1(G)$ .

- (iii) Take  $K = \mu B^H A$ .
- (iv) Take  $g_0 > 0$  and  $l_0 > 0$ .
- (v) For any  $i \in \mathcal{V}, j \in \mathcal{N}_i$ , take

$$\begin{aligned} L_{ji}(t) &= \max \left\{ \frac{\|C\|_\infty C_x}{g_0}, \frac{\|C\|_\infty}{g_0} \left( \frac{\tilde{H}M_7}{\gamma^{T_1}\gamma^{T_1+\tau_2}(\eta_0 - \gamma)(\gamma^{T_1+\tau_2} - \rho_2)} \right. \right. \\ &\quad \left. \left. + \frac{(1 + M_1R_0)(\|G\|g_0\sqrt{p}M_1 + \|B\|l_0\sqrt{m}M_1)}{(2\gamma^{T_1+\tau_2}(\gamma - \eta_1)\gamma^{T_1+\tau_2} - \rho_2)} \right) \right\} \\ &\quad - \frac{1}{2}, t \in \mathbb{N}, \\ L_{u,ji}(1) &= M_7 - \frac{1}{2}, \\ L_{u,ji}(t) &= \begin{cases} 2(L_{u,ij}(t-1) + \frac{1}{2}) - \frac{1}{2} & a_{ij}(t) = 0 \\ M_7 - \frac{1}{2} & a_{ij}(t) = 1 \end{cases}, t \geq 2, \end{aligned}$$

where  $M_7$  is given in Lemma A.2.



Then, under the communication and control protocol (9)–(11), the closed-loop system achieves synchronization exponentially fast.

For proving Theorem 3.1, we need the following lemmas.

*Lemma 3.4*

Under Assumption **A4**, if  $\tau_1 > n - 1$ , then the set

$$\left\{ X \in \mathbb{R}^{n \times p} \mid \eta_1(X) < \min \left\{ 1, (1/M_1(X)R_0)^{1/\tau_1} \right\} \right\}$$

is nonempty.

*Proof*

Because  $\varepsilon_1(X) = \frac{\rho(A-XC)}{\|A-XC\|}$ , we have

$$\eta_1(X) = 2\rho(A - XC), \quad M_1(X) = \sqrt{n} \left( 1 + \frac{2\|A - XC\|}{\rho(A - XC)} \right)^{n-1}.$$

So  $M_1(X)\eta_1^{\tau_1}(X)R_0 \leq 2^{\tau_1} \sqrt{n} R_0 \max \left\{ 2^{n-1}, \frac{(4\|A-XC\|)^{n-1}}{(\rho(A-XC))^{n-1}} \right\} (\rho(A-XC))^{\tau_1}$ . According to the pole assignment algorithm [19], there exist a series of  $X_j, j = 1, 2, \dots$ , such that  $\|X_j\|$  is uniformly upper bounded by a finite constant  $\bar{L}$  and  $\lim_{j \rightarrow \infty} \rho(A - X_j C) = 0$ . So when  $\tau_1 > n - 1$ , we have  $\lim_{j \rightarrow \infty} M_1(X_j)\eta_1^{\tau_1}(X_j)R_0 = 0$ . Thus, there exists  $X \in \mathbb{R}^{n \times p}$  such that  $\eta_1(X) < 1$  and  $M_1(X)\eta_1^{\tau_1}(X)R_0 < 1$ , which is equivalent to  $\eta_1(X) < (1/M_1(X)R_0)^{1/\tau_1}$ .  $\square$

*Lemma 3.5*

Under the assumptions in Theorem 3.1, the set  $\left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in \left( 0, \min_{i \in \mathcal{D}} \frac{1}{\|\mathcal{L}_i \otimes (A^H B B^H A)\|} \right), y \in \left( \max \left\{ 1/2, \eta_1, \rho_2^{1/(T_1+\tau_2)}, \beta(x) \right\}, 1 \right), a(x, y) < 1, b(x, y) < 1 \right\}$  is nonempty.

*Proof*

By some direct calculations, we have

$$\begin{aligned} \lim_{x \rightarrow 0} a(x, 1) &= \frac{2(1 - \eta_1)\tilde{H} \left( 4\sqrt{nN}d^* \|B^H A\|_\infty \|B B^H A\| x^2 + 2d^* \|B^H A\|_\infty x(1 - \beta(x)) \right) C_0(x)}{l_0(1 - \rho_2)(\eta_0 - 1)(1 - \eta_1)(1 - \beta(x))} \\ &= \lim_{x \rightarrow 0} \frac{8\sqrt{nN}d^* \|B^H A\|_\infty \|B B^H A\| (1 - \eta_1)\tilde{H}}{l_0(1 - \rho_2)(\eta_0 - 1)(1 - \eta_1)} \cdot \frac{x^2}{1 - \beta(x)}. \end{aligned} \tag{15}$$

By the L'Hospital's rule, it can be proved that  $\lim_{x \rightarrow 0} \frac{x^2}{1 - \beta(x)} = 0$ , thus  $\lim_{x \rightarrow 0} a(x, 1) = 0$ . Similarly, we have  $\lim_{x \rightarrow 0} b(x, 1) = 0$ , so there is an  $x^* \in (0, \min_{i \in \mathcal{D}} 1/\|\mathcal{L}_i \otimes (A^H B B^H A)\|)$  such that  $a(x^*, 1) < 1$  and  $b(x^*, 1) < 1$ . Noting that  $a(x^*, y)$  and  $b(x^*, y)$  are continuous functions of  $y$  at  $y = 1$ , thus there exists  $y^* \in \left( \max \left\{ 1/2, \eta_1, \rho_2^{1/(T_1+\tau_2)}, \beta(x^*) \right\}, 1 \right)$  such that  $a(x^*, y^*) < 1, b(x^*, y^*) < 1$ . So the set  $\left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in \left( 0, \min_{i \in \mathcal{D}} 1/\|\mathcal{L}_i \otimes (A^H B B^H A)\| \right), y \in \left( \max \left\{ 1/2, \eta_1, \rho_2^{1/(T_1+\tau_2)}, \beta(x) \right\}, 1 \right), a(x, y) < 1, b(x, y) < 1 \right\}$  is nonempty.  $\square$

*Remark 6*

Assumption **A4** assumes that the agent dynamics is observable. By this assumption, the poles of the state estimation error equation can be made sufficiently small to ensure the inter-agent state estimation errors to decrease sufficiently fast when the associate communication channels are active. If it is only assumed that  $(A, C)$  is detectable, then the minimum dwell time  $\tau_1$  of all communication channels should be very large to offset the divergence of the state estimation errors during the inactive periods of communication channels. It is an interesting topic how to balance the observability of the agent dynamics and the length of the dwell time of communication channels.

*Remark 7*

In Theorem 3.1, the parameters of the communication and control protocols are selected from some given parameter sets. From Lemmas 3.4 and 3.5, one can see that both the parameter sets in Theorem 3.1 are non-empty.

*Remark 8*

By Theorem 3.1, the parameter selection is divided into five steps with the main complexity in steps 1 and 2. As mentioned in Lemma 3.4, the pole assignment algorithm in [19] is helpful for selecting  $G$ . For selecting  $\mu$  and  $\gamma$  in step 2, one can first assign  $\gamma$  as 1 and then solve a second-order inequality of  $\mu$ . Then, by solving a similar inequality of  $\gamma$ , one can obtain the solution (see the proof of Lemma 3.5). In real applications, the parameter selection can be fulfilled by MATLAB tool boxes. A numerical example is given in Section 4 to verify the correctness and effectiveness of our protocols. Since the pole assignment algorithm in [19] is deeply related to the dimensions of the related matrices, the computation complexity may become large as the dimension of agents' state increases. It is an important issue how to reduce the computation complexity for the parameter selection. However, this is beyond the scope of this paper and is for future investigation.

*Proof of Theorem 3.1*

Denote

$$X(l) = (x_1^H(l), \dots, x_N^H(l))^H, U(l) = (u_1^H(l), \dots, u_N^H(l))^H, l \in \mathbb{N}.$$

By (8), we have  $X(l + 1) = (I_N \otimes A)X(l) + (I_N \otimes B)U(l), l \in \mathbb{N}$ . Denote

$$\begin{aligned} a_i(l) &= (a_{1i}(l), \dots, a_{Ni}(l))^H, \alpha_i(l) = (a_{i1}(l), \dots, a_{iN}(l))^H, \\ \Sigma_1(l) &= \text{diag}(a_1^H(l), \dots, a_N^H(l)), \Sigma_2(l) = \text{diag}(\alpha_1^H(l), \dots, \alpha_N^H(l)), \\ \hat{E}(l) &= (E_{11}^H(l), E_{21}^H(l), \dots, E_{N1}^H(l), E_{12}^H(l), \dots, E_{NN}^H(l))^H, \\ \bar{E}(l) &= (E_{11}^H(l), E_{12}^H(l), \dots, E_{1N}^H(l), E_{21}^H(l), \dots, E_{NN}^H(l))^H, l \in \mathbb{N}. \end{aligned}$$

Then, by (11), and noting that  $a_{ij}(l) = a_{ji}(l), i, j \in \mathcal{V}, l \in \mathbb{N}$ , we have  $U(l) = (-\mathcal{L}(l) \otimes K)X(l) + (\Sigma_1(l) \otimes K)\hat{E}(l) - (\Sigma_2(l) \otimes K)\bar{E}(l)$ , which leads to

$$\begin{aligned} X(l + 1) &= (I_N \otimes A - \mathcal{L}(l) \otimes BK)X(l) \\ &\quad + (\Sigma_1(l) \otimes BK)\hat{E}(l) - (\Sigma_2(l) \otimes BK)\bar{E}(l). \end{aligned} \tag{16}$$

Because  $a_{ij}(l) = a_{ji}(l)$ , it can be seen that  $(\mathbf{1}_N^H \otimes I_n)[(\Sigma_1(l) \otimes BK)\hat{E}(l) - (\Sigma_2(l) \otimes BK)\bar{E}(l)] = 0$ . This, together with  $\mathbf{1}_N^H \mathcal{L}(l) = 0$ , leads to  $(\mathbf{1}_N^H \otimes I_n)X(l + 1) = A(\mathbf{1}_N^H \otimes I_n)X(l)$ . Then, by (16), we have

$$\begin{aligned} \delta(l) &= X(l) - \frac{1}{N}(\mathbf{1}_N \mathbf{1}_N^H \otimes I_n)X(l) \\ &= (I_N \otimes A - \mathcal{L}(l - 1) \otimes BK)\delta(l - 1) \\ &\quad + (\Sigma_1(l - 1) \otimes BK)\hat{E}(l - 1) - (\Sigma_2(l - 1) \otimes BK)\bar{E}(l - 1) \\ &= \prod_{i=0}^{l-1} (I_N \otimes A - \mathcal{L}(i) \otimes BK)\delta(0) \\ &\quad + \sum_{i=0}^{l-1} \left[ \prod_{j=i+1}^{l-1} (I_N \otimes A - \mathcal{L}(j) \otimes BK) \left( (\Sigma_1(i) \otimes BK)\hat{E}(i) \right. \right. \\ &\quad \left. \left. - (\Sigma_2(i) \otimes BK)\bar{E}(i) \right) \right], l \in \mathbb{N}, \end{aligned} \tag{17}$$

where  $\prod_{j=l}^{l-1} (I_N \otimes A - \mathcal{L}(j) \otimes BK)$  is defined as  $I_{nN}$ . Denote  $z(l) = \prod_{i=0}^{l-1} (I_N \otimes A - \mathcal{L}(i) \otimes BK)\delta(0)$ ,  $l \geq 1$ . Let  $z(0) = \delta(0)$ , thus  $z(l) = (I_N \otimes A - \mathcal{L}(l-1) \otimes BK)z(l-1)$ , and  $(\mathbf{1}_N^H \otimes I_n)z(0) = 0, l \in \mathbb{N}$ . Then, by Lemma 3.3, one can see that

$$\|z(l)\| \leq C_0(\mu)\|z(0)\|\beta^l(\mu) = C_0(\mu)\|\delta(0)\|\beta^l(\mu), l \in \mathbb{N}. \tag{18}$$

For a positive integer  $i < l - 1$ , let  $v(l) = \prod_{j=i+1}^{l-1} (I_N \otimes A - \mathcal{L}(j) \otimes BK)[(\Sigma_1(i) \otimes BK)\hat{E}(i) - (\Sigma_2(i) \otimes BK)\bar{E}(i)]$ ,  $l > i + 1$  and let  $v(i + 1) = (\Sigma_1(i) \otimes BK)\hat{E}(i) - (\Sigma_2(i) \otimes BK)\bar{E}(i)$ , then we have

$$v(l) = (I_N \otimes A - \mathcal{L}(l-1) \otimes BK)v(l-1), l > i + 1.$$

Because  $(\mathbf{1}_N^H \otimes I_n) [(\Sigma_1(i) \otimes BK)\hat{E}(i) - (\Sigma_2(i) \otimes BK)\bar{E}(i)] = 0$ , by Lemma 3.3, we know that

$$\|v(l)\| \leq 2\sqrt{nN}d^*\beta^{l-1-i}(\mu)C_0(\mu)\|BK\| \max_{j \in \mathcal{V}, i \in \mathcal{N}_j} \|E_{ij}(i)\|. \tag{19}$$

From (17), (18), and (19), we have

$$\begin{aligned} \|\delta(l)\| &\leq \beta^l(\mu)C_0(\mu)\|\delta(0)\| \\ &+ \sum_{i=0}^{l-1} 2d^*\sqrt{nN}\|BK\|\beta^{l-1-i}(\mu)C_0(\mu) \max_{j \in \mathcal{V}, i \in \mathcal{N}_j} \|E_{ij}(i)\|, l \in \mathbb{N}. \end{aligned} \tag{20}$$

From (9) and (8), we have

$$\begin{aligned} E_{ij}(0) &= -x_i(0), \\ E_{ij}(1) &= (A - a_{ij}(1)GC)E_{ij}(0) + a_{ij}(1)g(0)G\Delta_{ij}(0), \end{aligned} \tag{21}$$

and

$$\begin{aligned} E_{ij}(t) &= \xi_{ij}(t) - x_i(t) \\ &= (A - a_{ij}(t)GC)E_{ij}(t-1) + a_{ij}(t)g(t-1)G\Delta_{ij}(t-1) \\ &\quad - (1 - a_{ij}(t))BW_{ij}(t-1) - a_{ij}(t)g_u(t-1)B\Delta_{u,ij}(t-1) \\ &= \prod_{k=0}^{t-1} (A - a_{ij}(k+1)GC)E_{ij}(0) \\ &\quad + \sum_{l=1}^{t-1} \prod_{k=l}^{t-1} (A - a_{ij}(k+1)GC)a_{ij}(l)g(l-1)G\Delta_{ij}(l-1) \\ &\quad - \sum_{l=2}^{t-1} \prod_{k=l}^{t-1} (A - a_{ij}(k+1)GC)(1 - a_{ij}(l))BW_{ij}(l-1) \\ &\quad - \sum_{l=2}^{t-1} \prod_{k=l}^{t-1} (A - a_{ij}(k+1)GC)a_{ij}(l)g_u(l-1)B\Delta_{u,ij}(l-1) \\ &\quad + a_{ij}(t)g(t-1)G\Delta_{ij}(t-1) - a_{ij}(t)g_u(t-1)B\Delta_{u,ij}(t-1) \\ &\quad - (1 - a_{ij}(t))BW_{ij}(t-1), t = 2, 3, \dots \end{aligned} \tag{22}$$

By Lemma A.2, we can see that the quantizers are not saturate at any time and

$$\begin{aligned} \|E_{ij}(l)\| &\leq \left( \frac{\tilde{H}M_7}{\gamma^{T_1}\gamma^{T_1+\tau_2}(\eta_0 - \gamma)(\gamma^{T_1+\tau_2} - \rho_2)} \right. \\ &\quad \left. + \frac{(1 + M_1R_0)(\|G\|g_0\sqrt{p}M_1 + \|B\|l_0\sqrt{m}M_1)}{2\gamma^{T_1+\tau_2}(\gamma - \eta_1)(\gamma^{T_1+\tau_2} - \rho_2)} \right) \cdot \gamma^l, l \in \mathbb{N}, j \in \mathcal{V}, i \in \mathcal{N}_j. \end{aligned} \tag{23}$$

This, together with (20) and the definition of  $M_2$ , leads to

$$\begin{aligned} \|\delta(l)\| \leq & 2d^* \sqrt{nN} \|BK\| C_0(\mu) \left( \tilde{H} M_7 / (\gamma^{T_1} \gamma^{T_1+\tau_2} (\eta_0 - \gamma) (\gamma^{T_1+\tau_2} - \rho_2) (\gamma - \beta(\mu))) \right. \\ & \left. + \frac{(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2\gamma^{T_1+\tau_2} (\gamma - \eta_1) (\gamma^{T_1+\tau_2} - \rho_2) (\gamma - \beta(\mu))} \right) \gamma^l = O(\gamma^l), \quad l \in \mathbb{N}. \end{aligned} \tag{24}$$

So the dynamic network  $(A, B, C, \mathcal{G}(t))$  achieves synchronization with an exponential speed  $\gamma$ .  $\square$

*Remark 9*

Su and Huang [17] studied the synchronization of linear multi-agent systems over jointly connected topologies with precise communication. Compared with [17], we considered the case with unmeasurable states and finite communication data rate. For achieving inter-agent state observation with output information and quantized communication, we propose the quantized observer-based communication protocol that integrates the inter-agent communication and state observation together. To overcome the information inconsistency between the sender and the receiver, we design the adaptive encoders and decoders given by (9) and (10), respectively, based on the channel activeness-based information updating rule. Furthermore, [17] did not give the convergence speed of the synchronization errors, while we show the exponential convergence speed of the synchronization errors with finite communication data rate.

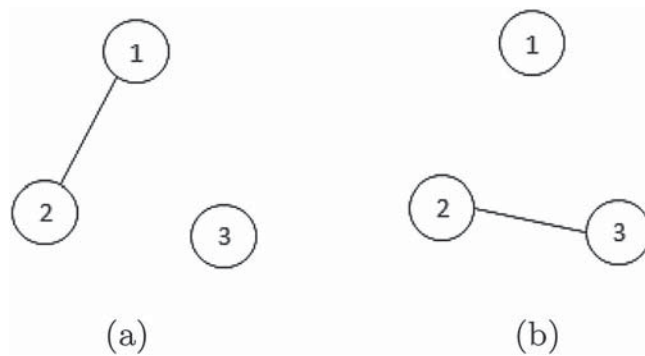


Figure 1. The communication topology flow.

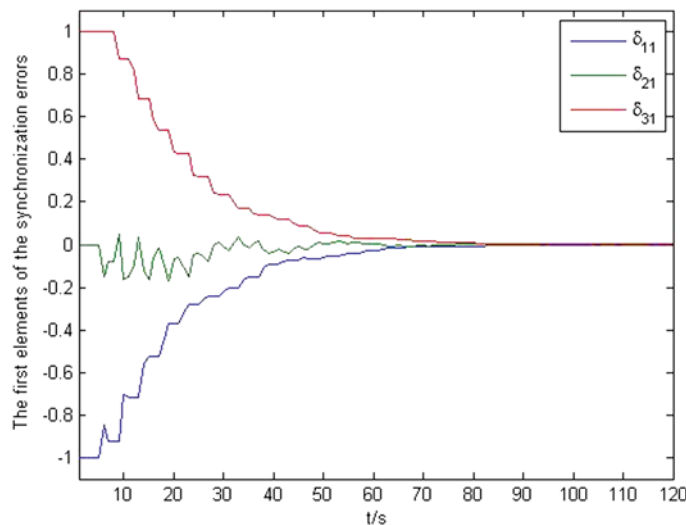


Figure 2. The first component of the synchronization error.

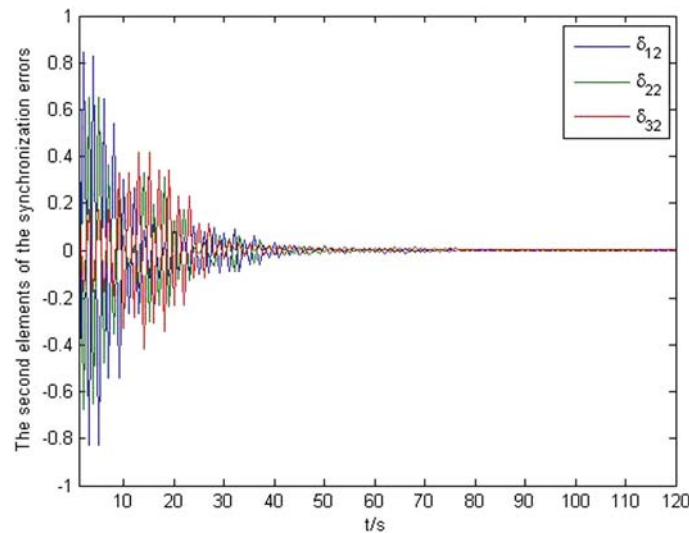


Figure 3. The second component of the synchronization error.

#### 4. NUMERICAL EXAMPLE

In this section, we give a numerical simulation example to show the effectiveness of our theoretical results. Here, we consider a multi-agent system with three agents, each having the following dynamics:

$$\begin{aligned} x_i(t+1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x_i(t) + \begin{pmatrix} 1 & 1 \\ -0.5 & -1.5 \end{pmatrix} u_i(t) \\ y_i(t) &= (1 \ 1) x_i(t) \end{aligned}$$

where  $i = 1, 2, 3$ . The inter-agent communication topology flow is a series of undirected graphs. The initial network topology is given by Figure 1(a). Then, the communication topology flow switches from Figure 1(b) to Figure 1(a) at  $t = 4k$ ,  $k = 1, 2, \dots$ , and switches from Figure 1(a) to Figure 1(b) at  $t = 4k + 2$ ,  $k = 0, 1, 2, \dots$ . The initial states of agents are selected randomly in  $(0, 3) \times (0, 3)$ .

It can be seen that Assumptions **A1–A6** hold and  $\tau_1 = 2 > n - 1 = 1$ . Here, we take  $G = \begin{pmatrix} 0.5 \\ -0.499 \end{pmatrix}$ ,  $\mu = 0.04$ ,  $\gamma = 0.95$ ,  $g_0 = l_0 = 1$ ,  $L_{ji}(t) \equiv 30$  for any  $i \in \mathcal{V}$ ,  $j \in \mathcal{N}_i$ ,  $t \in \mathbb{N}$  and  $M_7 = 50$ . The trajectories of two components of the synchronization errors are given in Figures 2 and 3. It is shown that the closed-loop system achieves synchronization under the proposed protocols.

#### 5. CONCLUSION

In this paper, we have studied the synchronization of linear multi-agent systems with unmeasurable states and uncertain network environments, especially, finite communication data rate and switching topologies. We proposed a class of adaptive quantized observer-based dynamic encoding–decoding schemes for information communication and a class of certainty equivalence principle-based control protocols for synchronization. By developing the graph-based space decomposition technique and analyzing the closed-loop quantized dynamic equations, it is shown that if the network topology flow is jointly connected, the communication channels are periodically active, and the agent dynamics is observable, and with the orthogonal system matrix, then the proposed protocols can drive the dynamic network to synchronization with finite bits of information exchange per step. For future research, it is of interest of developing the efficient numerical parameter selection algorithm and finding the lowest bound of the communication data rate to guarantee the synchronization.

APPENDIX

Lemma A.1 ([16])

For positive semi-definite matrices  $G_1$  and  $G_2$ , there is an orthogonal matrix  $U$ , such that

$$U^H(G_1 + G_2)U = \begin{pmatrix} G_{1,2} & 0 \\ 0 & 0 \end{pmatrix}$$

with

$$U^H G_1 U = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \text{ and } U^H G_2 U = \begin{pmatrix} \tilde{G}_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where  $\Lambda \in \mathbb{R}^{r_1 \times r_1}$  is a diagonal matrix with positive diagonal elements,  $G_{1,2} \in \mathbb{R}^{r_{1,2} \times r_{1,2}}$  is a positive definite matrix, and  $\tilde{G}_2 \in \mathbb{R}^{r_{1,2} \times r_{1,2}}$  is a positive semi-definite matrix.

Proof of Lemma 3.2

For proving Lemma 3.2, we first give some notations and a proposition, which specifies the exponential decay of the synchronization errors. Then, by this proposition and some direct calculations, we complete the proof of Lemma 3.2. Define a sequence of numbers

$$\gamma_{l+1} = (1 - \sigma^2)(1 - \gamma_l) + \gamma_l, l = 0, \dots, \tag{A.1}$$

where  $\gamma_0 = 1 - \mu \lambda \lambda_{min}$  and  $0 < \sigma < 1$  is the same as in Lemma 3.3. By (A.1), we can see that  $0 < \gamma_l < 1$  and  $\gamma_l < \gamma_{l+1}, l \in \mathbb{N}$ .

Consider any given interval  $[t_k, t_{k+1})$  with  $m_k$  subintervals. The Laplacian matrix associated with  $[t_{k_{i-1}}, t_{k_i})$  is  $\mathcal{L}_i, i = 1, \dots, m_k$ . Denote  $\mathcal{L}_{\widehat{j-1}} = \sum_{i=1}^{j-1} \mathcal{L}_i$  as the Laplacian matrix of the union graph on  $[t_k, t_{k_{j-1}})$ . Denote the rank of  $\mathcal{L}_{\widehat{j}}$  as  $r_{\widehat{j}}$ . For  $[t_k, t_{k_{j-1}})$  and  $[t_{k_{j-1}}, t_{k_j})$ , we have two matrices  $\mathcal{L}_{\widehat{j-1}}$  and  $\mathcal{L}_j$ . Then,  $\mathcal{L}_{\widehat{j}} = \mathcal{L}_{\widehat{j-1}} + \mathcal{L}_j$  is the Laplacian matrix of the union graph over  $[t_k, t_{k_j})$ .

Take  $G_1 = \mathcal{L}_{\widehat{j-1}}$  and  $G_2 = \mathcal{L}_j$ , then according to Lemma A.1, there is an orthogonal matrix  $U_1 = (\mathbf{e}_1^1, \dots, \mathbf{e}_N^1) \in \mathbb{R}^{N \times N}$  such that we can obtain three subspaces, which are denoted by

- $S_{\widehat{j},0}$  denotes the kernel of  $\mathcal{L}_{\widehat{j}}$  with  $\{\mathbf{e}_{r_{\widehat{j}+1}}^1, \dots, \mathbf{e}_N^1\}$  as its orthogonal basis.
- $S_{\widehat{j-1},\perp}$  with  $\{\mathbf{e}_1^1, \dots, \mathbf{e}_{r_{\widehat{j-1}}}^1\}$  as its basis of the subspace spanned by the eigenvectors corresponding to nonzero eigenvalues of  $\mathcal{L}_{\widehat{j-1}}$ .
- $S_{\widehat{j-1},-}$  with  $\{\mathbf{e}_{r_{\widehat{j-1}+1}}^1, \dots, \mathbf{e}_{r_{\widehat{j}}}^1\}$  as its orthogonal basis.

Denote  $S_{\widehat{j},\perp} = S_{\widehat{j-1},\perp} \oplus S_{\widehat{j-1},-}$ , which is the orthogonal complement of  $S_{\widehat{j},0}$ . Moreover, we have a transformation of  $\delta(t)$ , denoted by  $\delta^1(t)$ , that is,

$$\delta^1(t) = (U_1^H \otimes I_n) \delta(t).$$

For a vector  $x$  with dimension  $n$ , denote  $\tilde{V}(x) = x^H x$ , then we define the Lyapunov functions and related vectors in the subspaces

$$V_{\widehat{j},0}(\delta) = V_{\widehat{j},0} = \sum_{i=r_{\widehat{j}+1}}^N \tilde{V}(\delta_i^1), v_{\widehat{j},0} = \sum_{i=r_{\widehat{j}+1}}^N \sqrt{\tilde{V}(\delta_i^1)} \mathbf{e}_i^1,$$

$$V_{\widehat{j-1},\perp}(\delta) = V_{\widehat{j-1},\perp} = \sum_{i=1}^{r_{\widehat{j-1}}} \tilde{V}(\delta_i^1), v_{\widehat{j-1},\perp} = \sum_{i=1}^{r_{\widehat{j-1}}} \sqrt{\tilde{V}(\delta_i^1)} \mathbf{e}_i^1,$$

$$V_{\widehat{j-1,-}}(\delta) = V_{\widehat{j-1,-}} = \sum_{i=r_{\widehat{j-1}}+1}^{\widehat{j}} \tilde{V}(\delta_i^1), \quad v_{\widehat{j-1,-}} = \sum_{i=r_{\widehat{j-1}}+1}^{\widehat{j}} \sqrt{\tilde{V}(\delta_i^1)} \mathbf{e}_i^1.$$

Furthermore, according to Lemma A.1 with  $G_1 = \mathcal{L}_{\widehat{j-2}}$  and  $G_2 = \mathcal{L}_{j-1}$ , there is another orthogonal matrix  $U^{1'} = (\mathbf{e}_1^{1'}, \dots, \mathbf{e}_N^{1'})$ , and  $(\mathbf{e}_{r_{\widehat{j-1}+1}}^{1'}, \dots, \mathbf{e}_N^{1'})$  as the basis of the kernel of  $\mathcal{L}_{\widehat{j-1}}$ . So we can take  $e_i^{1'} = e_i^1, i = r_{\widehat{j-1}+1}, \dots, N$  [16]. Then, it follows that

$$V_{\widehat{j-1,0}} = \sum_{i=r_{\widehat{j-1}}+1}^N \tilde{V}(\delta_i^1).$$

On the other hand, taking  $G_1 = \mathcal{L}_j$  and  $G_2 = \mathcal{L}_{\widehat{j-1}}$  in Lemma A.1, there is another orthogonal matrix  $U_2 = (\mathbf{e}_1^2, \dots, \mathbf{e}_N^2)$  and a new transformation of  $\delta(t)$  as  $\delta^2(t) = (U_2^H \otimes I_n)\delta(t)$ . Also, we have three subspaces, still with the first subspace  $S_{\widehat{j},0}$  as the kernel of  $\mathcal{L}_{\widehat{j}}$ . Thus, we can take  $\mathbf{e}_i^1 = \mathbf{e}_i^2$  for  $i = r_{\widehat{j}} + 1, \dots, N$  [16]. The other two spaces are as follows:

- $S_{j,\perp}$ , with its orthogonal basis  $\{\mathbf{e}_1^2, \dots, \mathbf{e}_{r_j}^2\}$ , which are the eigenvectors associated with the nonzero eigenvalues of  $\mathcal{L}_j$ .
- $S_{j,-}$  is a subspaces with  $\{\mathbf{e}_{r_j+1}^2, \dots, \mathbf{e}_{r_{\widehat{j}}}^2\}$  as its orthogonal basis.

Similarly, define  $V_{j,\perp}(\delta) = \sum_{i=1}^{r_j} \tilde{V}(\delta_i^2), v_{j,\perp} = \sum_{i=1}^{r_j} \sqrt{\tilde{V}(\delta_i^2)} \mathbf{e}_i^2$  and  $V_{j,-} = \sum_{i=r_j+1}^{\widehat{j}} \tilde{V}(\delta_i^2), v_{j,-} = \sum_{i=r_j+1}^{\widehat{j}} \sqrt{\tilde{V}(\delta_i^2)} \mathbf{e}_i^2$ . Obviously,

$$S_{\widehat{j-1,\perp}} \oplus S_{\widehat{j-1,-}} = S_{j,\perp} \oplus S_{j,-} = S_{\widehat{j},\perp} \tag{A.2}$$

because they have the same orthogonal complement space  $S_{\widehat{j},0}$ . It can be easily proved that [16]

$$S_{\widehat{j-1,-}} \cap S_{j,-} = \{0\}. \tag{A.3}$$

Therefore, by their definitions, we can see that  $V = V_{\widehat{j},0} + V_{\widehat{j-1,\perp}} + V_{\widehat{j-1,-}} = V_{\widehat{j},0} + V_{j,\perp} + V_{j,-}$ . For convenience, we denote

$$V_{\widehat{j},\perp}(\delta) = V_{\widehat{j-1,\perp}}(\delta) + V_{\widehat{j-1,-}}(\delta) = V_{j,\perp}(\delta) + V_{j,-}(\delta), \tag{A.4}$$

and, correspondingly, the vector  $v_{\widehat{j},\perp} = v_{\widehat{j-1,\perp}} + v_{\widehat{j-1,-}} = v_{j,\perp} + v_{j,-}$  with

$$\|v_{\widehat{j},\perp}\| = \sqrt{\|v_{\widehat{j-1,\perp}}\|^2 + \|v_{\widehat{j-1,-}}\|^2} = \sqrt{\|v_{j,\perp}\|^2 + \|v_{j,-}\|^2}. \tag{A.5}$$

For continuing the proof, we need the following proposition. □

*Proposition A.1*

Suppose that there is a  $j \geq 2$  such that

$$V_{\widehat{j-1,0}}(\delta(t_{k_{j-1}})) \leq V_{\widehat{j-1,0}}(\delta(t_k)) \tag{A.6}$$

and

$$V_{\widehat{j-1,\perp}}(\delta(t_{k_{j-1}})) \leq \gamma_{j-2} V_{\widehat{j-1,\perp}}(\delta(t_k)), \tag{A.7}$$

where  $0 < \gamma_{j-2} < 1$  is defined in (A.1). Then, (A.6) and (A.7) hold with  $\gamma_{j-1}$ , that is,

$$V_{\widehat{j},0}(\delta(t_{k_j})) \leq V_{\widehat{j},0}(\delta(t_k)) \tag{A.8}$$

and

$$V_{j,\perp}(\delta(t_{k_j})) \leq \gamma_{j-1} V_{j,\perp}(\delta(t_k)). \tag{A.9}$$

The proof of this proposition is the same as Lemma 5 in [16] and is omitted here.

Now, we give the proof of Lemma 3.2. Consider an interval  $[t_k, t_{k+1})$  consisting of a sequence of subintervals  $[t_{k_{j-1}}, t_{k_j}), j = 1, \dots, m_k, t_{k_j} - t_{k_{j-1}} \geq 1, t_{k_0} = t_k, t_{k_{m_k}} = t_{k+1}$ . The topology flow does not switch during each subinterval. Let  $\xi = \gamma_T$  where  $\gamma_i, i = 1, \dots, T$ , are defined in (A.1).

Consider the first subinterval  $[t_k, t_{k_1})$ . According to Remark 2, we can see that (A.8) and (A.9) hold for  $j = 1$ . Assuming (A.8) and (A.9) hold for  $j = p - 1$ , then by Proposition A.1, we can see that (A.8) and (A.9) hold for  $j = p$ . By induction, we know that (A.8) and (A.9) hold for all  $j = 1, \dots, m_k$ . Because  $\mathcal{L}_1, \dots, \mathcal{L}_{m_k}$  are jointly connected over  $[t_k, t_{k+1})$ ,  $\widehat{\mathcal{L}}_{m_k}$  has only one zero eigenvalue. Thus,  $V_{m_k,\perp}(\delta) = \sum_{i=1}^{r_{m_k}} (\delta_i^1)^H \delta_i^1 = \sum_{i=1}^{N-1} (\delta_i^1)^H \delta_i^1$  where  $\left( (\delta_1^1)^H, \dots, (\delta_N^1)^H \right)^H = \delta^1 = (U_1^H \otimes I_n) \delta$  and  $U_1 = (\mathbf{e}_1^1, \dots, \mathbf{e}_N^1)$  is the orthogonal matrix defined in Lemma A.1 with  $\mathcal{L}_{m_k-1} = G_1$  and  $\mathcal{L}_{m_k} = G_2$ . By Lemma A.1, it can be easily seen that  $\mathbf{e}_N^1 = \frac{1}{\sqrt{N}} \mathbf{1}_N$ ; thus,  $\delta_N^1 = \left( (\mathbf{e}_N^1)^H \otimes I_n \right) \delta = 0$ . So,

$$V_{m_k,\perp}(\delta) = \sum_{i=1}^{N-1} (\delta_i^1)^H \delta_i^1 = \sum_{i=1}^N (\delta_i^1)^H \delta_i^1 = \sum_{i=1}^N (\delta_i)^H \delta_i = V(\delta). \tag{A.10}$$

Noting that  $\xi = \gamma_T \geq \gamma_{m_k-1}$ , then from Proposition A.1 and (A.10), we have  $V(\delta(t_{k+1})) = V(\delta(t_{m_k})) = V_{m_k,\perp}(\delta(t_{m_k})) \leq \gamma_{m_k-1} V_{m_k,\perp}(\delta(t_k)) \leq \xi V(\delta(t_k))$ . Thus, for any  $t \in \mathbb{N}$ , we have  $V(\delta(t)) \leq \xi^{t/T-1} V(\delta(0))$ . Denote  $\beta = \xi^{1/2T}, C_0 = \frac{1}{\sqrt{\xi}}$ . Then, we have  $\|\delta(t)\| \leq \beta^t C_0 \|\delta(0)\|$ .

*Lemma A.2*

Suppose that the conditions of Theorem 3.1 are satisfied. By selecting the parameters as in Theorem 3.1, where  $M_7 = \max\{M_2, M_3, M_4, M_5, M_6\}$ ,

$$\begin{aligned} M_2 &= (C_\delta \gamma^{T_1} \gamma^{T_1+\tau_2} (\gamma - \eta_1)(\eta_0 - \gamma)(\gamma - \beta(\mu)) (\gamma^{T_1+\tau_2} - \rho_2) \\ &\quad - d^* \mu \gamma^{T_1} \|BB^H A\| (\eta_0 - \gamma)(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1) / \\ &\quad (2d^* \mu \|BB^H A\| \tilde{H}(\gamma - \eta_1)), \\ M_3 &= 2d^* \mu \|B^H A\|_\infty \left( \|A\|_\infty C_x + \left( \max\{\|A - GC\|, \|A\|\} \sqrt{n} C_x + \frac{g_0 \sqrt{p} \|G\|}{2} \right) \right) / l_0 \gamma, \\ M_4 &= (2\sqrt{n} C_x \gamma^{T_1} (\eta_0 - \gamma) (\gamma^{T_1+\tau_2} - \rho_2) (\gamma - \eta_1) \\ &\quad - (\eta_0 - \gamma) \|G\| g_0 \sqrt{p} M_1 + (\eta_0 - \gamma) \|B\| l_0 \sqrt{m} M_1) / (2^{T_1+1} \rho_2 l_0 \sqrt{m} \|B\| (\gamma - \eta_1)), \\ M_5 &= (\gamma^{T_1} (\eta_0 - \gamma)(\gamma + 1) H(\mu, \gamma)(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1) \\ &\quad + l_0 \gamma^{T_1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2) (\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1) / \\ &\quad (2l_0 \gamma^{T_1+1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1) - 2(\gamma + 1)(\gamma - \eta_1) H(\mu, \gamma) \tilde{H})), \\ M_6 &= (\gamma^{T_1+1} (\gamma + 1)(\eta_0 - \gamma) H(\mu, \gamma)(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1) / \\ &\quad (2(2l_0 \gamma - l_0) \gamma^{T_1+1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1) \\ &\quad - \gamma(\gamma + 1)(\gamma - \eta_1) H(\mu, \gamma) \tilde{H})), \end{aligned}$$

then, the quantizers will never be saturate, and

$$\begin{aligned} \|E_{ij}(l)\| &\leq \left( \frac{\tilde{H} M_7}{\gamma^{T_1} \gamma^{T_1+\tau_2} (\eta_0 - \gamma) (\gamma^{T_1+\tau_2} - \rho_2)} \right. \\ &\quad \left. + \frac{(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2\gamma^{T_1+\tau_2} (\gamma - \eta_1) (\gamma^{T_1+\tau_2} - \rho_2)} \right) \cdot \gamma^l, \quad i \in \mathcal{N}_j, j \in \mathcal{V}, l \in \mathbb{N}. \end{aligned}$$



*Proof*

We will use mathematical induction to obtain the conclusion. First consider the initial time. Because  $\| \frac{y_i(0) - C\xi_{ij}(0)}{g(0)} \|_\infty = \| \frac{CE_{ij}(0)}{g_0} \|_\infty \leq \frac{\|C\|_\infty C_x}{g_0} \leq L_{ij}(0) + \frac{1}{2}$ , we can see at the initial time,  $Q_0^{ij}(\cdot)$  is not saturate and  $\| \Delta_{ij}(0) \|_\infty \leq \frac{1}{2}$ ,  $i \in \mathcal{N}_j, j \in \mathcal{V}$ . By (9), we have

$$\frac{u_i(1) - \hat{u}_{ij}(0)}{g_u(1)} = \frac{u_i(1)}{l_0\gamma} \tag{A.11}$$

From (11), we know that

$$\begin{aligned} u_i(1) &= K \sum_{j=1}^N a_{ij}(1)(\hat{x}_{ji}(1) - \xi_{ij}(1)) \\ &= K \sum_{j=1}^N a_{ij}(1)(x_j(1) - x_i(1)) + K \sum_{j=1}^N a_{ij}(1)E_{ji}(1) - K \sum_{j=1}^N a_{ij}(1)E_{ij}(1). \end{aligned} \tag{A.12}$$

By (8), we have  $x_i(1) = Ax_i(0) + Bu_i(0) = Ax_i(0)$ ; thus, by (A.12), we have

$$\begin{aligned} \| u_i(1) \|_\infty &\leq 2 \| K \|_\infty d^* \| A \|_\infty C_x \\ &\quad + 2 \| K \|_\infty d^* \max_{i,j} \| E_{ij}(1) \|_\infty. \end{aligned} \tag{A.13}$$

According to (22), if  $a_{ij}(1) = 1$ , then  $E_{ij}(1) = (A - GC)E_{ij}(0) + g_0G\Delta_{ij}(0)$ , and if  $a_{ij}(1) = 0$ , then  $E_{ij}(1) = AE_{ij}(0)$ , so  $\|E_{ij}(1)\|_\infty \leq \|E_{ij}(1)\| \leq \max\{\|A - GC\|, \|A\|\} \|E_{ij}(0)\| + g_0\|G\| \|\Delta_{ij}(0)\| \leq \max\{\|A - GC\|, \|A\|\} \sqrt{n}C_x + \frac{g_0\sqrt{p}\|G\|}{2}$ . This, together with (A.13) and the definition of  $M_3$ , leads to  $\| \frac{u_i(1)}{l_0\gamma} \|_\infty \leq \frac{g_0 d^* \sqrt{p} \|G\| \|K\|_\infty}{l_0\gamma} + \frac{2\|K\|_\infty d^* \|A\|_\infty C_x}{l_0\gamma} + \frac{2\|K\|_\infty d^* \max\{\|A - GC\|, \|A\|\} \sqrt{n}C_x}{l_0\gamma} \leq L_{u,ij}(1) + \frac{1}{2}$ . Noting that for the quantizer  $Q_k^{u,ij}(\cdot)$ , its initial time is  $k = 1$ . So at the initial time  $Q_1^{u,ij}(\cdot)$  is also not saturate,  $i \in \mathcal{N}_j, j \in \mathcal{V}$ .

Assume that the quantizers are not saturate before time  $t$ , that is,  $\max_{i,j} \| \Delta_{ij}(k) \|_\infty \leq \frac{1}{2}$ ,  $0 \leq k \leq t - 1$  and  $\max_{ij} \| \Delta_{u,ij}(k) \|_\infty \leq \frac{1}{2}$ ,  $1 \leq k \leq t - 1$ . Because  $Q_k^{u,ij}(\cdot)$  are not saturate at  $k = 1, \dots, t - 1$ , noting **A5** and the selection of  $L_{u,ij}(k)$ , we have

$$\left\| \frac{W_{ij}(k)}{g_u(k)} \right\|_\infty \leq L_{u,ij}(k) + \frac{1}{2} \leq 2^{T_1} M_7, \quad k = 1, \dots, t - 1. \tag{A.14}$$

Now, we consider time  $t$ . For time  $t$  and any channel  $(i, j)$ , there exist some  $k$  such that  $t \in [t_k^{ij}, t_{k+1}^{ij})$ . Without loss of generality, let  $t = t_{k+1}^{ij} - 1$ , because other cases can be analyzed similarly. If  $(i, j)$  is not active on  $[t_k^{ij}, t_{k+1}^{ij})$ , then  $a_{ij}(l) = 0, l = t_k^{ij}, \dots, t_{k+1}^{ij} - 1$ . By **A5**, it can be seen that  $t_{k+1}^{ij} - t_k^{ij} = T_1^{ij}$ . Thus, from (22), we have

$$E_{ij}(l) = AE_{ij}(l - 1) - BW_{ij}(l - 1), \quad l = t_k^{ij}, \dots, t_{k+1}^{ij} - 1. \tag{A.15}$$

Because  $Q_k^{u,ij}(\cdot)$  is not saturate before  $t = t_{k+1}^{ij} - 1$ , from (A.14), we have  $\|W_{ij}(l)\| \leq \sqrt{m} \|W_{ij}(l)\|_\infty \leq \sqrt{m} 2^{T_1} M_7 l_0 \gamma^l, l = 1, \dots, t_{k+1}^{ij} - 2$ . By (A.15), we can see that

$$\begin{aligned} E_{ij}(t_{k+1}^{ij} - 1) &= A^{t_{k+1}^{ij} - t_k^{ij}} E_{ij}(t_k^{ij} - 1) \\ &\quad - \sum_{l=t_k^{ij} - 1}^{t_{k+1}^{ij} - 2} A^{t_{k+1}^{ij} - 2 - l} B W_{ij}(l), \end{aligned} \tag{A.16}$$

which together with Lemma 2.4.1 of [20] and the definition of  $M_0$  and  $\eta_0$  leads to

$$\begin{aligned} \|E_{ij}(t_{k+1}^{ij} - 1)\| &\leq M_0 \eta_0^{T_1} \|E_{ij}(t_k^{ij} - 1)\| \\ &\quad + \frac{\sqrt{m} \|B\| 2^{T_1} l_0 M_0 \eta_0^{T_1} M_7}{\gamma^{T_1} (\eta_0 - \gamma)} \cdot \gamma^{t_{k+1}^{ij}-1}. \end{aligned} \tag{A.17}$$

By the definition of  $R_0$  and (A.17), we know that

$$\|E_{ij}(t_{k+1}^{ij} - 1)\| \leq R_0 \|E_{ij}(t_k^{ij} - 1)\| + \frac{\sqrt{m} \|B\| 2^{T_1} l_0 R_0 M_7}{\gamma^{T_1} (\eta_0 - \gamma)} \cdot \gamma^{t_{k+1}^{ij}-1}. \tag{A.18}$$

Similarly, one can prove that for any  $l \in [t_k^{ij}, t_{k+1}^{ij})$ , if  $(i, j)$  is inactive on this interval, then we have

$$\|E_{ij}(l)\| \leq R_0 \|E_{ij}(t_k^{ij} - 1)\| + \frac{\sqrt{m} \|B\| 2^{T_1} l_0 R_0 M_7}{\gamma^{T_1} (\eta_0 - \gamma)} \cdot \gamma^l. \tag{A.19}$$

If  $(i, j)$  is active on  $[t_k^{ij}, t_{k+1}^{ij})$ , then by **A5**, we know that  $t_{k+1}^{ij} - t_k^{ij} = \tau_1^{ij}$  and  $a_{ij}(l) = 1, l = t_k^{ij}, \dots, t_{k+1}^{ij} - 1$ . By (22), we know that

$$\begin{aligned} E_{ij}(l) &= (A - GC)E_{ij}(l - 1) + g(l - 1)G\Delta_{ij}(l - 1) \\ &\quad - g_u(l - 1)B\Delta_{u,ij}(l - 1), \quad l = t_k^{ij}, \dots, t_{k+1}^{ij} - 1. \end{aligned}$$

Thus,

$$\begin{aligned} E_{ij}(t_{k+1}^{ij} - 1) &= (A - GC)^{t_{k+1}^{ij}-t_k^{ij}} E_{ij}(t_k^{ij} - 1) + \sum_{l=t_k^{ij}-1}^{t_{k+1}^{ij}-2} (A - GC)^{t_{k+1}^{ij}-2-l} G g_0 \gamma^l \Delta_{ij}(l) \\ &\quad - \sum_{l=t_k^{ij}-1}^{t_{k+1}^{ij}-2} (A - GC)^{t_{k+1}^{ij}-2-l} B l_0 \gamma^l \Delta_{u,ij}(l). \end{aligned} \tag{A.20}$$

Because  $Q_k^{ij}(\cdot)$  and  $Q_k^{u,ij}(\cdot)$  are not saturate before  $t = t_{k+1}^{ij} - 1, j \in \mathcal{V}, i \in \mathcal{N}_j$ , by (A.20), Lemma 2.4.1 of [20] and the definition of  $M_1$  and  $\eta_1$ , we know that

$$\begin{aligned} \|E_{ij}(t_{k+1}^{ij} - 1)\| &\leq \eta_1^{t_{k+1}^{ij}-t_k^{ij}} \|E_{ij}(t_k^{ij} - 1)\| + \|G\| g_0 \frac{\sqrt{p}}{2} M_1 \sum_{l=t_k^{ij}-1}^{t_{k+1}^{ij}-2} \eta_1^{t_{k+1}^{ij}-2-l} \gamma^l \\ &\quad + \|B\| l_0 \frac{\sqrt{m}}{2} M_1 \sum_{l=t_k^{ij}-1}^{t_{k+1}^{ij}-2} \eta_1^{t_{k+1}^{ij}-2-l} \gamma^l \\ &\leq M_1 \eta_1^{\tau_1^{ij}} \|E_{ij}(t_k^{ij} - 1)\| \\ &\quad + \frac{(\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2(\gamma - \eta_1)} \cdot \gamma^{t_{k+1}^{ij}-1}. \end{aligned} \tag{A.21}$$

Similarly, one can prove that for any  $l \in [t_k^{ij}, t_{k+1}^{ij})$ , if  $(i, j)$  is active on this interval, then we have

$$\|E_{ij}(l)\| \leq M_1 \|E_{ij}(t_k^{ij} - 1)\| + \frac{(\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2(\gamma - \eta_1)} \cdot \gamma^l. \tag{A.22}$$

Now, we prove that quantizers  $Q^{ij}(\cdot)$  and  $Q^{u,ij}(\cdot)$ ,  $i \in \mathcal{N}_j$ ,  $j \in \mathcal{V}$ , are not saturate at time  $t$ . For any  $l \in \{1, 2, \dots, k-1\}$ , if  $(i, j)$  is inactive on  $[t_{l-1}^{ij}, t_l^{ij})$ , then it is active on  $[t_l^{ij}, t_{l+1}^{ij})$ . Thus, from **A5**, we know that  $t_l^{ij} - t_{l-1}^{ij} = T_1^{ij}$  and  $t_{l+1}^{ij} - t_l^{ij} = \tau_1^{ij}$ . Let  $[\alpha_h^{ij}, \alpha_{h+1}^{ij}) = [t_{l-1}^{ij}, t_l^{ij}) \cup [t_l^{ij}, t_{l+1}^{ij})$ . Following (A.18) and (A.21), we have

$$\begin{aligned} \|E_{ij}(\alpha_{h+1}^{ij} - 1)\| &\leq M_1 \eta_1^{\tau_1} R_0 \|E_{ij}(\alpha_h^{ij} - 1)\| \\ &\quad + \left( \frac{2^{T_1} M_1 R_0 l_0 \sqrt{m} \|B\| M_7}{\gamma^{T_1} (\eta_0 - \gamma)} + \frac{\|G\| \sqrt{p} g_0 M_1 + \|B\| l_0 \sqrt{m} M_1}{2(\gamma - \eta_1)} \right) \cdot \gamma^{\alpha_{h+1}^{ij} - 1}. \end{aligned} \tag{A.23}$$

Without loss of generality, we assume that  $(i, j)$  is inactive on  $[0, t_1^{ij})$ . Because  $\alpha_{h+1}^{ij} - \alpha_h^{ij} = T_1^{ij} + \tau_1^{ij}$ ,  $h \in \mathbb{N}$ , if  $(i, j)$  is active on  $[t_k^{ij}, t_{k+1}^{ij})$ , there are  $\frac{t}{T_1^{ij} + \tau_1^{ij}}$  intervals  $[\alpha_h^{ij}, \alpha_{h+1}^{ij})$  in  $[0, t]$ . Thus, noting that  $t = t_{k+1}^{ij} - 1$  and (A.23), if  $(i, j)$  is active on  $[t_k^{ij}, t_{k+1}^{ij})$ , we have

$$\begin{aligned} \|E_{ij}(t)\| &= \|E_{ij}(t_{k+1}^{ij} - 1)\| \\ &\leq \rho_2^{\frac{t_{k+1}^{ij} - 1}{T_1^{ij} + \tau_1^{ij}}} \|E_{ij}(0)\| + \left( \frac{\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1}{2(\gamma - \eta_1)} \right. \\ &\quad \left. + \frac{2^{T_1} \rho_2 l_0 \sqrt{m} \|B\| M_7}{\gamma^{T_1} (\eta_0 - \gamma)} \right) \sum_{i=1}^{\frac{t_{k+1}^{ij} - 1}{T_1^{ij} + \tau_1^{ij}}} \rho_2^{\frac{t_{k+1}^{ij} - 1}{T_1^{ij} + \tau_1^{ij}} - i} \gamma^{(T_1^{ij} + \tau_1^{ij})i} \\ &\leq \rho_2^{\frac{t_{k+1}^{ij} - 1}{T_1^{ij} + \tau_1^{ij}}} \|E_{ij}(0)\| + \left( \frac{2^{T_1} \gamma^{T_1^{ij} + \tau_1^{ij}} \rho_2 l_0 \sqrt{m} \|B\| M_7}{\gamma^{T_1} (\eta_0 - \gamma) (\gamma^{T_1 + \tau_2} - \rho_2)} \right. \\ &\quad \left. + \frac{\gamma^{T_1^{ij} + \tau_1^{ij}} \|G\| g_0 \sqrt{p} M_1 + \gamma^{T_1^{ij} + \tau_1^{ij}} \|B\| l_0 \sqrt{m} M_1}{2(\gamma - \eta_1) (\gamma^{T_1 + \tau_2} - \rho_2)} \right) \\ &\quad \times \left( \gamma^{t_{k+1}^{ij} - 1} - \rho_2^{\frac{t_{k+1}^{ij} - 1}{T_1^{ij} + \tau_1^{ij}}} \right). \end{aligned} \tag{A.24}$$

Because  $M_7 \geq M_4$  and  $\max_{i \in \mathcal{N}_j, j \in \mathcal{V}} \|E_{ij}(0)\| \leq \sqrt{n} C_x$ , then by (A.24), we have

$$\begin{aligned} \|E_{ij}(t)\| &= \|E_{ij}(t_{k+1}^{ij} - 1)\| \\ &\leq \left( \frac{2^{T_1} \rho_2 l_0 \sqrt{m} \|B\| M_7}{\gamma^{T_1} (\eta_0 - \gamma) (\gamma^{T_1 + \tau_2} - \rho_2)} + \frac{\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1}{2(\gamma - \eta_1) (\gamma^{T_1 + \tau_2} - \rho_2)} \right) \cdot \gamma^{t_{k+1}^{ij} - 1}. \end{aligned} \tag{A.25}$$

Noting (A.19) and (A.22), it can be similarly proved that for any  $l \leq t$ , there is

$$\begin{aligned} \|E_{ij}(l)\| &\leq \left( \frac{\tilde{H} M_7}{\gamma^{T_1} \gamma^{T_1 + \tau_2} (\eta_0 - \gamma) (\gamma^{T_1 + \tau_2} - \rho_2)} \right. \\ &\quad \left. + \frac{(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2\gamma^{T_1 + \tau_2} (\gamma - \eta_1) (\gamma^{T_1 + \tau_2} - \rho_2)} \right) \gamma^l, \end{aligned} \tag{A.26}$$

no matter whether  $(i, j)$  is active or not on  $[t_k^{ij}, t_{k+1}^{ij})$ . By (A.25) and (A.26), one can see that

$$\begin{aligned} \frac{\|CE_{ij}(t)\|_\infty}{g(t)} &\leq \frac{\|C\|_\infty}{g_0} \left( \frac{\tilde{H}M_7}{\gamma^{T_1}\gamma^{T_1+\tau_2}(\eta_0-\gamma)(\gamma^{T_1+\tau_2}-\rho_2)} \right. \\ &\quad \left. + \frac{(1+M_1R_0)(\|G\|g_0\sqrt{p}M_1+\|B\|l_0\sqrt{m}M_1)}{2\gamma^{T_1+\tau_2}(\gamma-\eta_1)(\gamma^{T_1+\tau_2}-\rho_2)} \right) \\ &< L_{ij}(t) + \frac{1}{2}. \end{aligned}$$

So  $Q_k^{ij}(\cdot)$  is not saturate at time  $t$ .

Next, we prove that  $Q_k^{u,ij}(\cdot)$  is also not saturate at  $t$ . From (11) and the definition of  $\delta(t)$ , we know that

$$\begin{aligned} u_i(l) &= K \sum_{j=1}^N a_{ij}(l)(x_j(l) - x_i(l)) + K \sum_{j=1}^N a_{ji}(l)E_{ji}(l) - K \sum_{j=1}^N a_{ij}(l)E_{ij}(l) \\ &= K \sum_{j=1}^N a_{ij}(l)(\delta_j(l) - \delta_i(l)) + K \sum_{j=1}^N a_{ji}(l)E_{ji}(l) - K \sum_{j=1}^N a_{ij}(l)E_{ij}(l), \quad l = 1, 2, \dots \end{aligned}$$

So

$$\|u_i(l)\|_\infty \leq 2d^* \|K\|_\infty \|\delta(l)\| + 2d^* \|K\|_\infty \max_{j \in \mathcal{V}, i \in \mathcal{N}_j} \|E_{ij}(l)\|. \tag{A.27}$$

Similarly, we have  $\|u_i(l-1)\|_\infty \leq 2d^* \|K\|_\infty \|\delta(l-1)\| + 2d^* \|K\|_\infty \max_{i \in \mathcal{N}_j, j \in \mathcal{V}} \|E_{ij}(l-1)\|$ . By (20), we can see that

$$\begin{aligned} \|\delta(l)\| &\leq \beta^l(\mu)C_0(\mu)\|\delta(0)\| \\ &\quad + \sum_{i=0}^{l-1} 2\sqrt{nN}d^* \|BK\| \beta^{l-1-i}(\mu)C_0(\mu) \max_{i \in \mathcal{N}_j, j \in \mathcal{V}} \|E_{ij}(i)\|, \quad l \in \mathbb{N}. \end{aligned} \tag{A.28}$$

This, together with (A.26), leads to

$$\begin{aligned} \|\delta(l)\| &\leq \beta^l(\mu)C_0(\mu)\|\delta(0)\| \\ &\quad + 2d^* \sqrt{nN} \|BK\| C_0(\mu) \left( \frac{\tilde{H}M_7}{\gamma^{T_1}\gamma^{T_1+\tau_2}(\eta_0-\gamma)(\gamma^{T_1+\tau_2}-\rho_2)} \right. \\ &\quad \left. + \frac{(1+M_1R_0)(\|G\|g_0\sqrt{p}M_1+\|B\|l_0\sqrt{m}M_1)}{2\gamma^{T_1+\tau_2}(\gamma-\eta_1)(\gamma^{T_1+\tau_2}-\rho_2)} \right) \cdot \sum_{i=0}^{l-1} \beta^{l-1-i}(\mu)\gamma^i \\ &\leq \beta^l(\mu)C_0(\mu)\|\delta(0)\| \\ &\quad + 2d^* \sqrt{nN} \|BK\| C_0(\mu) \left( \frac{\tilde{H}M_7}{\gamma^{T_1}\gamma^{T_1+\tau_2}(\eta_0-\gamma)(\gamma^{T_1+\tau_2}-\rho_2)(\gamma-\beta(\mu))} \right. \\ &\quad \left. + \frac{(1+M_1R_0)(\|G\|g_0\sqrt{p}M_1+\|B\|l_0\sqrt{m}M_1)}{2\gamma^{T_1+\tau_2}(\gamma-\eta_1)(\gamma^{T_1+\tau_2}-\rho_2)(\gamma-\beta(\mu))} \right) \cdot (\gamma^l - \beta^l(\mu)), \\ l &\leq t_{k+1}^{ij} - 1. \end{aligned} \tag{A.29}$$

Because  $M_7 \geq M_2$ , then from (A.29), it is known that

$$\begin{aligned} \|\delta(l)\| \leq & 2d^* \sqrt{nN} \|BK\| C_0(\mu) \left( \tilde{H} M_7 / (\gamma^{T_1} \gamma^{T_1+\tau_2} (\eta_0 - \gamma)(\gamma^{T_1+\tau_2} - \rho_2)(\gamma - \beta(\mu))) \right. \\ & \left. + \frac{(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2\gamma^{T_1+\tau_2} (\gamma - \eta_1)(\gamma^{T_1+\tau_2} - \rho_2)(\gamma - \beta(\mu))} \right) \gamma^l, \quad \forall 0 < l \leq t_{k+1}^{ij} - 1. \end{aligned} \tag{A.30}$$

Denote  $f(\mu, \gamma) = \left( \frac{\tilde{H} M_7}{\gamma^{T_1} \gamma^{T_1+\tau_2} (\eta_0 - \gamma)(\gamma^{T_1+\tau_2} - \rho_2)(\gamma - \beta(\mu))} + \frac{(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2\gamma^{T_1+\tau_2} (\gamma - \eta_1)(\gamma^{T_1+\tau_2} - \rho_2)(\gamma - \beta(\mu))} \right)$ , then from (A.27), (A.30), and (A.25), we know that

$$\begin{aligned} \|u_i(t_{k+1}^{ij} - 1)\| &= \|u_i(t)\| \\ &\leq 4d^* \sqrt{nN} \|K\|_\infty \|BK\| C_0(\mu) f(\mu, \gamma) \gamma^t \\ &\quad + 2d^* \|K\|_\infty (\gamma - \beta(\mu)) C_0(\mu) f(\mu, \gamma) \gamma^t \\ &= C_0(\mu) \left( 4d^* \sqrt{nN} \|K\|_\infty \|BK\| \right. \\ &\quad \left. + 2d^* \|K\|_\infty (\gamma - \beta(\mu)) \right) f(\mu, \gamma) \gamma^t. \end{aligned} \tag{A.31}$$

Noting that  $K = \mu B^H A$ , by the definition of  $H(\mu, \gamma)$ , we have  $H(\mu, \gamma) = 4d^* \sqrt{nN} \|K\|_\infty \|BK\| C_0(\mu) + 2d^* \|K\|_\infty (\gamma - \beta(\mu)) C_0(\mu)$ . Thus, by (A.31), we know that  $\|u_i(t)\|_\infty \leq H(\mu, \gamma) f(\mu, \gamma) \gamma^t$ . With the same method, we have  $\|u_i(t - 1)\|_\infty \leq H(\mu, \gamma) f(\mu, \gamma) \gamma^{t-1}$ . Now, consider the input of quantizer  $Q_k^{u,ij}(\cdot)$  at time  $t$ .

If  $a_{ij}(t) = 1$ , then by (9), we know that

$$u_i(t) - \hat{u}_{ij}(t - 1) = u_i(t) - u_i(t - 1) - g_u(t - 1) \Delta_{u,ij}(t - 1).$$

Noting that  $\|u_i(t)\|_\infty \leq H(\mu, \gamma) f(\mu, \gamma) \gamma^t$ ,  $\|u_i(t - 1)\|_\infty \leq H(\mu, \gamma) f(\mu, \gamma) \gamma^{t-1}$ , and  $Q_k^{u,ij}(\cdot)$  is not saturate before time  $t$ , we have

$$\begin{aligned} & \frac{\|u_i(t) - \hat{u}_{ij}(t - 1)\|_\infty}{g_u(t)} \\ & \leq \frac{(\gamma + 1) H(\mu, \gamma) f(\mu, \gamma)}{\gamma l_0} + \frac{1}{2\gamma} \\ & = \frac{(\gamma + 1) H(\mu, \gamma) \tilde{H} M_7}{\gamma l_0 \gamma^{T_1} \gamma^{T_1+\tau_2} (\eta_0 - \gamma) (\gamma^{T_1+\tau_2} - \rho_2) (\gamma - \beta(\mu))} \\ & \quad + \frac{(\gamma + 1) H(\mu, \gamma) (1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2\gamma l_0 \gamma^{T_1+\tau_2} (\gamma - \eta_1) (\gamma^{T_1+\tau_2} - \rho_2) (\gamma - \beta(\mu))} + \frac{1}{2\gamma} \\ & = \frac{(\gamma + 1) (\gamma - \eta_1) H(\mu, \gamma) \tilde{H} M_7}{l_0 \gamma^{T_1+1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2) (\eta_0 - \gamma) (\gamma - \beta(\mu)) (\gamma - \eta_1)} \\ & \quad + \frac{\gamma^{T_1} (\eta_0 - \gamma) (\gamma + 1) H(\mu, \gamma) (1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{2l_0 \gamma^{T_1+1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2) (\eta_0 - \gamma) (\gamma - \beta(\mu)) (\gamma - \eta_1)} \\ & \quad + \frac{l_0 \gamma^{T_1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2) (\eta_0 - \gamma) (\gamma - \beta(\mu)) (\gamma - \eta_1)}{2l_0 \gamma^{T_1+1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2) (\eta_0 - \gamma) (\gamma - \beta(\mu)) (\gamma - \eta_1)}. \end{aligned} \tag{A.32}$$

By the selection of  $\mu, \gamma$  and Lemma 3.5, we have  $(\gamma + 1) (\gamma - \eta_1) H(\mu, \gamma) \tilde{H} < l_0 \gamma^{T_1+1} \gamma^{T_1+\tau_2} (\gamma^{T_1+\tau_2} - \rho_2) (\eta_0 - \gamma) (\gamma - \rho_1) (\gamma - \eta_1)$ . Because  $M_7 \geq M_5$ , from (A.32), we know that  $\frac{\|u_i(t) - \hat{u}_{ij}(t - 1)\|_\infty}{g_u(t)} < M_7 = L_{u,ij}(t) + \frac{1}{2}$ . Thus, if  $a_{ij}(t) = 1$ , the quantizer  $Q_k^{u,ij}(\cdot)$  is not saturate at time  $t$ .

If  $a_{ij}(t) = 0$ , then by (9), we know that

$$\begin{aligned} \frac{\|u_i(t) - \hat{u}_{ij}(t-1)\|_\infty}{g_u(t)} &= \frac{\|u_i(t) - \hat{u}_{ij}(t-2)\|_\infty}{g_u(t)} \\ &\leq \frac{\|u_i(t) - u_i(t-1)\|_\infty + \|W_{ij}(t-1)\|_\infty}{g_u(t)}. \end{aligned} \tag{A.33}$$

Because  $Q_k^{u,ij}(\cdot)$  is not saturate before  $t$ , from (A.33), we have

$$\begin{aligned} &\frac{\|u_i(t) - \hat{u}_{ij}(t-1)\|_\infty}{g_u(t)} \\ &\leq \frac{(\gamma + 1)H(\mu, \gamma)f(\mu, \gamma)}{l_0\gamma} + \frac{L_{u,ij}(t-1) + \frac{1}{2}}{\gamma} \\ &= \frac{(\gamma + 1)(\gamma - \eta_1)H(\mu, \gamma)\tilde{H}M_7}{l_0\gamma^{T_1+1}\gamma^{T_1+\tau_2}(\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1)} \\ &\quad + \frac{\gamma^{T_1}(\eta_0 - \gamma)(\gamma + 1)H(\mu, \gamma)(1 + M_1R_0)(\|G\|g_0\sqrt{p}M_1 + \|B\|l_0\sqrt{m}M_1)}{2l_0\gamma^{T_1+1}\gamma^{T_1+\tau_2}(\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1)} \\ &\quad + \frac{l_0L_{u,ij}(t-1) + \frac{l_0}{2}}{l_0\gamma}. \end{aligned} \tag{A.34}$$

By Lemma 3.5, we have  $(\gamma + 1)(\gamma - \eta_1)H(\mu, \gamma)\tilde{H} < (2l_0\gamma - l_0)\gamma^{T_1}\gamma^{T_1+\tau_2}(\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1)$ . By the selection of quantization levels in Theorem 3.1, we know that  $M_7 \leq L_{u,ij}(l) + \frac{1}{2}$  for any  $l \in \mathbb{N}$ . What's more, because  $M_7 \geq M_6$ , we know that

$$\begin{aligned} &\frac{(\gamma + 1)(\gamma - \eta_1)H'(\mu, \gamma)\tilde{H}M_7}{l_0\gamma^{T_1+1}\gamma^{T_1+\tau_2}(\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1)} \\ &\quad + \frac{\gamma^{T_1}(\eta_0 - \gamma)(\gamma + 1)H'(\mu, \gamma)(1 + M_1R_0)(\|G\|g_0\sqrt{p}M_1 + \|B\|l_0\sqrt{m}M_1)}{2l_0\gamma^{T_1+1}\gamma^{T_1+\tau_2}(\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1)} \\ &\leq \left(2 - \frac{1}{\gamma}\right)M_7 \leq \left(2 - \frac{1}{\gamma}\right)\left(L_{u,ij}(t-1) + \frac{1}{2}\right). \end{aligned} \tag{A.35}$$

From (A.35) and (A.34), we have

$$\begin{aligned} &\frac{\|u_i(t) - \hat{u}_{ij}(t-1)\|_\infty}{g_u(t)} \\ &\leq \frac{(\gamma + 1)(\gamma - \eta_1)H(\mu, \gamma)\tilde{H}M_7}{l_0\gamma^{T_1+1}\gamma^{T_1+\tau_2}(\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1)} \\ &\quad + \frac{\gamma^{T_1}(\eta_0 - \gamma)(\gamma + 1)H(\mu, \gamma)(1 + M_1R_0)(\|G\|g_0\sqrt{p}M_1 + \|B\|l_0\sqrt{m}M_1)}{l_0\gamma^{T_1+1}\gamma^{T_1+\tau_2}(\gamma^{T_1+\tau_2} - \rho_2)(\eta_0 - \gamma)(\gamma - \beta(\mu))(\gamma - \eta_1)} \\ &\quad + \frac{L_{u,ij}(t-1) + \frac{1}{2}}{\gamma} \\ &\leq \left(2 - \frac{1}{\gamma}\right)\left(L_{u,ij}(t-1) + \frac{1}{2}\right) + \frac{L_{u,ij}(t-1) + \frac{1}{2}}{\gamma} \\ &= \left(L_{u,ij}(t-1) + \frac{1}{2}\right) \\ &= L_{u,ij}(t) + \frac{1}{2}. \end{aligned}$$

So if  $a_{ij}(t) = 0$ , the quantizer  $Q_k^{u,ij}(\cdot)$  is not saturate at  $t$ . From the aforementioned discussion, we know that  $Q_k^{u,ij}(\cdot)$  is not saturate at  $t$ . Thus, by induction, the quantizers  $Q_k^{ij}(\cdot)$  and  $Q_k^{u,ij}(\cdot)$ ,  $j \in$

$\mathcal{V}, i \in \mathcal{N}_j$  are not saturate at any time  $t \in \mathbb{N}$ . By (A.26), and noting that the quantizers are not saturate at any time, it can be easily proved that

$$\|E_{ij}(l)\| \leq \left( \frac{\tilde{H} M_7}{\gamma^{T_1} \gamma^{T_1 + \tau_2} (\eta_0 - \gamma) (\gamma^{T_1 + \tau_2} - \rho_2)} + \frac{(1 + M_1 R_0) (\|G\| g_0 \sqrt{p} M_1 + \|B\| l_0 \sqrt{m} M_1)}{(2\gamma^{T_1 + \tau_2} (\gamma - \eta_1) \gamma^{T_1 + \tau_2} - \rho_2)} \right) \gamma^l, l \in \mathbb{N}.$$

□

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#### REFERENCES

1. Qu ZH. *Cooperative Control of Dynamic System*. Springer: London, 2009.
2. Olfati-Saber R, Murray RM. Consensus problem in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* 2004; **49**(9):1520–1533.
3. Fax JA, Murray RM. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control* 2004; **49**(9):1465–1476.
4. Ren W, Beard RW. *Distributed Consensus in Multi-Vehicle Cooperative Control*. Springer: London, 2008.
5. Frasca P, Carli R, Fagnani F, Zampieri S. Average consensus on networks with quantized communication. *International Journal of Robust and Nonlinear Control* 2009; **19**(16):1787–1816.
6. Carli R, Bullo F, Zampieri S. Quantized average consensus via dynamic coding/decoding schemes. *International Journal of Robust and Nonlinear Control* 2010; **20**(2):156–175.
7. Li T, Fu M, Xie L, Zhang JF. Distributed consensus with limited communication data rate. *IEEE Transactions on Automatic Control* 2011; **56**(2):279–292.
8. Li D, Liu Q, Wang X, Lin Z. Consensus seeking over directed networks with limited information communication. *Automatica* 2013; **49**(11):3329–3336.
9. Li T, Xie L. Distributed consensus over digital networks with limited bandwidth and time-varying topologies. *Automatica* 2011; **47**(9):2006–2015.
10. Zhang Q, Zhang JF. Quantized data-based distributed consensus under directed time-varying communication topology. *SIAM Journal on Control and Optimization* col. 2013; **51**(1):332–352.
11. Li D, Liu Q, Wang X, Lin Z. Quantized Consensus over directed networks with switching topologies. *System & Control Letters* 2014; **45**:13–22.
12. Ma CQ, Zhang JF. Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control* 2010; **55**(5):1263–1268.
13. You K, Xie L. Network topology and communication data rate for consensusability of discrete-time multi-agent systems. *IEEE Transactions on Automatic Control* 2011; **56**(10):2262–2275.
14. Hengster-Movric K, You K, Lewis FL, Xie L. Synchronization of discrete-time multi-agent systems on graphs using Riccati design. *Automatica* 2013; **49**(2):414–423.
15. Wen G, Ugrinovskii V. Distributed  $H_\infty$  tracking control for discrete-time multi-agent systems with a high-dimensional leader. *Proceedings of the 52nd IEEE Conference on Decision and Control*, Florence, Italy, Winter, 10–13 December 2013; 2163–2168.
16. Hong YG, Gao LX, Cheng DZ, Hu JP. Lyapunov-based approach to multiagent systems with switching jointly connected interconnection. *IEEE Transactions on Automatic Control* 2007; **52**(5):943–948.
17. Su YF, Huang J. Two consensus problems for discrete-time multi-agent systems with switching network topology. *Automatica* 2012; **48**(9):1988–1997.
18. Li T, Xie L. Distributed coordination of multi-agent systems with quantized-observer based encoding-decoding. *IEEE Transactions on Automatic Control* 2012; **57**(12):3023–3037.
19. Zheng DZ. *Linear System Theory*. TsingHua University Press: Beijing, 2005.
20. Guo L, System—Stability Time-varying Stochastic. *Estimation and Control*. Jilin Science and Tecnology Press: Changchun, 1993.